



FACULDADE DE MATEMÁTICAS

Undergraduate Dissertation

# Dirac Operators

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Degree in Mathematics  
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# Trabajo propuesto

<b>Área de Conocimiento:</b> Geometría y Topología
<b>Título:</b> Operadores de Dirac
<b>Breve descripción del contenido</b>  <ol style="list-style-type: none"><li>1. Se empezará con una introducción a las variedades spin y geometría spin, con énfasis en los ejemplos.</li><li>2. También se hará una introducción a operadores diferenciales en variedades.</li><li>3. Se definirá el operador de Dirac y se estudiarán sus propiedades básicas.</li><li>4. Se culminará con el estudio espectral del operador de Dirac en variedades compactas. Se harán algunos cálculos concretos. Si llegase el tiempo, se probarían cotas inferiores del primer autovalor no nulo.</li></ol>
<b>Recomendaciones</b>  Es conveniente cursar la materia Variedades Diferenciables. En el caso de estudiantes del doble grado de Matemáticas y Física, es interesante cursar la materia Teoría Cuántica de Campos.
<b>Otras observaciones</b>  Esta propuesta de TFG es adecuada para ser complementada con alguna propuesta de TFG del Grado en Física. El operador de Dirac juega un papel importante en Mecánica Cuántica.



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## Abstract

This work is a mostly self-contained survey on Dirac operators. It starts by laying the fundamental building blocks at the heart of spin geometry, specifically its elemental geometrical and algebraic aspects. Thus, the concepts of vector and principal bundles over manifolds, Clifford algebras, the Pin and Spin groups, the spin representation and the spinor bundle are explored. A brief commentary on connections and linear differential operators on manifolds is also provided. Subsequently, the fundamental Dirac operator is presented, along with a review of its most important basic properties. The last section is devoted to a study of the Dirac spectrum on compact manifolds, including some explicit computations and bounds of the lower nonzero eigenvalue.

**Keywords**— fiber bundles; connections; Clifford algebras; Spin group; spin representation; spinor bundle; Dirac operator; Dirac spectrum

## Resumen

Este trabajo es un estudio esencialmente autocontenido de los operadores de Dirac. Empieza por establecer las bases fundamentales de la geometría espinorial, específicamente sus aspectos geométricos y algebraicos elementales. Así, se exploran los conceptos de fibrados vectoriales y principales, así como las álgebras de Clifford, los grupos Pin y Spin, la representación espinorial y el fibrado de espinores. También se proporciona un breve comentario sobre conexiones y operadores diferenciales en variedades. A continuación, se presenta el operador fundamental de Dirac, junto con un repaso a sus propiedades básicas más importantes. La última sección está dedicada al estudio espectral del operador de Dirac en variedades compactas, incluyendo algunos cálculos concretos y estimaciones de las cotas inferiores del primer autovalor no nulo.

**Palabras clave**— fibrados; conexiones; álgebras de Clifford; grupo espinorial; representación espinorial; fibrado de espinores; operador de Dirac; espectro de Dirac



# Preface

The Dirac operator owes its name to English physicist Paul Dirac, who discovered it during his studies on the wave operator within the context of quantum mechanics and its relation to general relativity [1]. It was originally envisioned as a means to describe the probability amplitude of fermions (particles of half-integer spin). Nonetheless, as is the case with many objects used by physicists, it soon drew attention from the mathematical community.

The first mathematically sound description of Dirac operators came from M. F. Atiyah and I. Singer, in their papers on the index of elliptic operators (for example, [2]). Their findings laid the groundwork for the birth of a whole new field of knowledge: spin geometry.

Any attempt to dive deep into the subtleties of spin geometry leads to numerous interrelations between topology, geometry and analysis. Therefore, the study of Dirac operators entails the need to explore some preliminary concepts belonging to very different fields, in an effort to bring them together to discuss the construction of such operators, and their implications within the framework of spin geometry.

This work starts by reviewing some of the essential building blocks needed for the study of Dirac operators. Namely, it explores the fundamentals of Clifford algebras, spin groups and spin representations, as well as notions of fiber bundles and Riemannian geometry (such as differential operators on manifolds). Then, it explicitly introduces the Dirac operator and analyzes some of its properties, including the explicit computations of its spectrum on some closed manifolds and estimations of the lower nonzero eigenvalue. No previous knowledge of spin geometry is required, although a basic background in linear algebra, differential geometry and topology is recommended.

A reader who is familiarized with spin geometry and differential operators on manifolds might wish to skip to chapter 3, where the analysis focuses on Dirac operators themselves. Otherwise, this work provides all the necessary tools for their definition and analysis.



# Chapter 1

## Preliminaries on differential geometry and global analysis

The study of Dirac operator involves concepts related to manifolds and vector bundles, as well as algebraic notions of algebras and linear representations. Since the purpose of this work is not to retread the treatment of these subjects from the ground up, the reader will only be introduced in a thorough manner to the essential tools for the task at hand. Refer to [3], [4] for a more in-depth look on differential geometry, and to books like [5] for an extensive review on algebraic topics.

### 1.1 Fiber bundles

#### 1.1.1 General fiber bundles

A *local trivialization*  $(U, \phi)$  with *typical fiber*  $F$  for a smooth surjective mapping  $\pi : E \rightarrow M$  is a diffeomorphism  $\phi : \pi^{-1}(U) \rightarrow U \times F$  such that the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi} & U \times F \\ & \searrow \pi & \swarrow p \\ & U & \end{array}$$

commutes, where  $p$  is the projection onto the first factor. A (*smooth*) *fiber bundle* with *typical fiber*  $F$  is a smooth map  $\pi : E \rightarrow M$ , together with a maximal family of local

trivializations  $\{U_i, \phi_i\}$  with typical fiber  $F$  so that  $\{U_i\}$  is an open cover of  $M$ . It is said that  $M$  is the *base space* and  $E$  is the *total space*; often, it is simply said that  $E$  is a fiber bundle over  $M$ . With this conditions,  $\pi$  becomes a surjective submersion. The *fiber* over  $x \in M$  is  $E_x := \pi^{-1}(x)$ , which is a regular submanifold of  $E$ . The fiber bundle is determined by any subfamily of trivializations that cover  $M$ .

Given two local trivializations,  $(U_i, \phi_i)$  and  $(U_j, \phi_j)$ , and some point  $x \in U_i \cap U_j$ , there exists a diffeomorphism  $g_{ij}(x) : F \rightarrow F$  given by the composition

$$F \equiv \{x\} \times F \xrightarrow{\phi_i^{-1}} E_x \xrightarrow{\phi_j} \{x\} \times F \equiv F.$$

This defines a map to the group of diffeomorphisms of  $F$ ,  $g_{ij} : U_i \cap U_j \rightarrow \text{Diffeo}(F)$ , which is *smooth* in the sense that the map  $(U_i \cap U_j) \times F \rightarrow F$ ,  $(x, u) \mapsto g_{ij}(x)(u)$ , is smooth. Given a family  $\{U_i, \phi_i\}$  of local trivializations defining the fiber bundle as above, the corresponding family of maps  $g_{ij}$  is called a *defining cocycle* of the fiber bundle, usually denoted by  $\{U_i, g_{ij}\}$ . It satisfies the *defining cocycle property*

$$g_{ij}(x) \circ g_{jk}(x) = g_{ik}(x) \quad (1.1.1)$$

for all  $x \in U_i \cap U_j \cap U_k$ ; in particular, one gets  $g_{ii}(x) = \text{id}_F$ . The defining cocycle  $\{U_i, g_{ij}\}$  describes the fiber bundle over  $M$ ; in fact,  $E \equiv (\bigsqcup_i U_i \times F) / \sim$ , where  $(i, x, u) \sim (j, x, g_{ij}(x)(u))$  for  $x \in U_i \cap U_j$  (this is an equivalence relation by (1.1.1)), and  $\pi : E \rightarrow M$  is induced by  $\bigsqcup_i U_i \times F \rightarrow M$ ,  $(i, x, u) \mapsto x$ . Thus a fiber bundle  $\pi : E \rightarrow M$  with typical fiber  $F$  can be given by a defining cocycle; i.e. a collection  $\{U_i, g_{ij}\}$ , consisting of an open covering  $\{U_i\}$  of  $M$ , and smooth maps  $g_{ij} : U_i \cap U_j \rightarrow \text{Diffeo}(F)$  satisfying (1.1.1).

Given a subgroup  $G \subset \text{Diffeo}(F)$ , a *fiber bundle* with (*typical fiber*  $F$  and) *structural group*  $G$  is a fiber bundle described by a (maximal) defining cocycle  $\{U_i, g_{ij}\}$  with values in  $G$ ; i.e., with  $g_{ij} : U_i \cap U_j \rightarrow G$  for all  $i, j$ . This concept will be used to easily define different types of fiber bundles. For another subgroup  $G \subset H \subset \text{Diffeo}(F)$ , any fiber bundle with structural group  $G$  becomes a fiber bundle with structural group  $H$ . The reverse direction may not be true; if a fiber bundle with structural group  $H$  admits a structure of fiber bundle with structural group  $G$ , it is said that its structural group  $H$  can be *reduced* to  $G$ .

Let  $f : N \rightarrow M$  be a smooth map between manifolds, and let  $E$  be a fiber bundle over  $M$  with defining cocycle  $\{U_i, g_{ij}\}$ . Then  $\{f^{-1}(U_i), g_{ij} \circ f\}$  is a defining cocycle over  $N$ , obtaining a fiber bundle over  $N$  with the same typical fiber, which is denoted by  $f^*E$  and called the *pull-back* of  $E$  by  $f$ . If  $E$  has a particular structural group, then  $f^*E$  has the same structural group.

An *isomorphism* of fiber bundles  $\pi_a : E_a \rightarrow M$  ( $a = 1, 2$ ) (with structural group  $G$ ) is a

diffeomorphism  $h : E_1 \rightarrow E_2$  such that  $\pi_1 = \pi_2 \circ h$  (so that the structural groups correspond via  $h$  in an obvious way). A fiber bundle  $\pi : E \rightarrow M$  with typical fiber  $F$  (and structural group  $G$ ) is said to be *trivial* if it is isomorphic to the fiber bundle (with structural group  $G$ ) given by the first factor projection  $M \times F \rightarrow M$ .

A (*smooth*) *section* of a fiber bundle  $\pi : E \rightarrow M$  is a smooth map  $s : M \rightarrow E$  such that  $\pi \circ s = \text{id}_M$ . The collection of all smooth sections of  $\pi : E \rightarrow M$  is denoted by  $\Gamma(E)$ . With more generality, a (*smooth*) *section* of  $\pi : E \rightarrow M$  over an open subset  $U \subset M$  is a smooth map  $s : U \rightarrow E$  so that  $\pi \circ s$  is the inclusion map  $U \hookrightarrow M$ . The collection of sections of  $\pi : E \rightarrow M$  over  $U$  is denoted by  $\Gamma(U, E)$ .

### 1.1.2 Vector bundles

A (*smooth*) *real vector bundle* of rank  $r \in \mathbb{N}$  is a (smooth) fiber bundle  $\pi : E \rightarrow M$  with typical fiber  $\mathbb{R}^r$  and structural group  $\text{GL}(r, \mathbb{R})$ . Then the real vector space structure of the typical fiber  $\mathbb{R}^r$  induces a real vector space structure on every fiber  $E_x$  in a canonical way. Thus the vector bundle can be considered as a smoothly varying family of real vector spaces  $E_x$  parametrized by the points  $x$  of  $M$ . In other words, a real vector bundle of rank  $r$  is a fiber bundle  $\pi : E \rightarrow M$  with typical fiber  $\mathbb{R}^r$ , equipped with a maximal collection of trivializations,  $\{U_i, \phi_i\}$ , so that every fiber  $E_x$  is a real vector space and  $\phi_i : E_x \rightarrow \{x\} \times \mathbb{R}^r \equiv \mathbb{R}^r$  is a linear isomorphism.

By using fiberwise scalar multiplication,  $\Gamma(E)$  becomes a module over the ring of real valued smooth functions on  $M$ ,  $C^\infty(M)$ . In particular, one always has the zero section  $0 \in \Gamma(E)$ , assigning the zero of the fiber  $E_x$  to every  $x \in M$ . The *support* of a section  $s$  of  $E$  is the closure  $\text{supp } s = \overline{\{x \in M : s(x) \neq 0\}}$ . The submodule of compactly supported sections is denoted by  $\Gamma_c(E)$ . Similarly,  $\Gamma_c(U, E)$  is used for compactly supported sections over an open subset  $U$ . The extension by zero defines a canonical injection  $\Gamma_c(U, E) \hookrightarrow \Gamma_c(V, E)$  for  $U \subset V$ ; in particular,  $\Gamma_c(U, E) \hookrightarrow \Gamma_c(E)$ .

*Isomorphisms* and *triviality* of vector bundles are particular cases of their analogues in general fiber bundles with a given structural group. If a vector bundle  $E$  of rank  $r$  over  $M$  is trivial, then  $\Gamma(E) \equiv C^\infty(M, \mathbb{R}^r)$ .

All functors of vector spaces, which preserve smoothness on the parameter when applied to smooth families of vector spaces or linear maps, have natural extensions to vector bundles over  $M$ . These extensions have simple definitions by using defining cocycles. For

example, let  $E, F$  be vector bundles over  $M$  with ranks  $r, q$ , and defining cocycles  $\{U_i, g_{ij}\}$  and  $\{U_i, h_{ij}\}$ , respectively (after refinement, the same open cover can be taken in both cocycles). Then  $E \oplus F$  is the vector bundle of rank  $r + q$  with defining cocycle  $\{U_i, k_{ij}\}$ , where  $k_{ij}(x) = g_{ij}(x) \oplus h_{ij}(x) \in \text{GL}(\mathbb{R}^r \oplus \mathbb{R}^s) \equiv \text{GL}(r + s, \mathbb{R})$ . Similarly, it is possible to define  $E \otimes F$ ,  $E^*$ ,  $\bigwedge E$ ,  $E \odot E$ , etc. Here, the notation  $\odot$  is used for the symmetric tensor product (see [16] for more information on  $\odot$ ). Note that  $E^* \odot E^*$  is the vector bundle whose fibers consist of the symmetric bilinear forms on the fibers of  $E$ .

**Example 1.1.1** (Tangent bundle). Let  $\{U_i, x_i\}$  be an atlas of a manifold  $M$  of dimension  $n$ . The *tangent space*  $TM$  is the vector bundle over  $M$  of rank  $n$  defined by the cocycle given by the differential of the changes of coordinates,  $D(x_j^{-1} \circ x_i) : U_i \cap U_j \rightarrow \text{GL}(n, \mathbb{R})$ ,  $p \mapsto D(x_j^{-1} \circ x_i)(x_i(p))$ . Its smooth sections are the (*tangent*) *vector fields*, and the notation  $\mathfrak{X}(M) = \Gamma(TM)$  is used.

An *Euclidean structure* of a vector bundle  $E$  of rank  $r$  is the choice of a positive definite scalar product  $g_x$  on every fiber  $E_x$  such that  $x \mapsto g_x$  is a smooth section  $g$  of  $E^* \odot E^*$ . If  $E$  is equipped with a Euclidean structure, then it is called a *Euclidean vector bundle*; this is the same as reducing its structural group to  $\text{O}(r)$ .

An *orientation* of a vector bundle  $E$  of rank  $r$  is given by a reduction of its structural group to  $\text{SL}(r, \mathbb{R})$ , or to  $\text{SO}(r)$  after equipping it with some Euclidean structure. Not every vector bundle is orientable; for instance, the orientability of  $TM$  as a vector bundle means the orientability of  $M$  as a manifold.

All of the above concepts have obvious complex versions, changing  $\mathbb{R}$  to  $\mathbb{C}$ . In this case, the conjugate bundle  $\overline{E}$  can also be considered, whose fibers are the conjugate vector spaces of the fibers of  $E$ . Then a *Hermitian structure* on a complex vector bundle  $E$  is the choice of a Hermitian product  $g_x$  on every fiber  $E_x$  defining a smooth section of  $E^* \odot \overline{E}^*$ . If  $E$  is equipped with a Hermitian structure, then it is called a *Hermitian vector bundle*; this is the same as reducing its structural group to  $\text{U}(r)$ .

Any real/complex vector bundle admits a Euclidean/Hermitian structure [4, pp. 73-76].

For any Euclidean/Hermitian vector bundle  $E$  on a Riemannian manifold  $M \equiv (M, g)$ , with Riemannian density  $\nu_g$ , there is a scalar product  $\langle \cdot, \cdot \rangle_{L^2}$  on  $\Gamma_c(E)$  defined by

$$\langle s_1, s_2 \rangle_{L^2} = \int_M \langle s_1(x), s_2(x) \rangle \nu_g(x).$$

The corresponding Hilbert space completion is denoted by  $L^2(E)$ .



### 1.1.3 Principal bundles

This is another major group of fiber bundles that will be of interest here. Let  $G$  be a Lie group. Its left translations form a subgroup of  $\text{Diffeo}(G)$  isomorphic to  $G$ . They can be characterized as the only transformations of  $G$  that are equivariant with respect to the right action on itself by right translations. A (*smooth*) *principal bundle* with *structural group*  $G$ , or a *principal  $G$ -bundle*, is a fiber bundle  $\pi : P \rightarrow M$  with typical fiber  $G$  whose structural group is the group of left translations on  $G$ . By the above characterization of left translations, it follows that there is a smooth free right action of  $G$  on  $P$  whose orbits are the fibers.

*Isomorphisms* and *triviality* of principal  $G$ -bundles are particular cases of their analogues for general fiber bundles with a given structural group. A principal  $G$ -bundle  $P$  is trivial if it has a section  $s : M \rightarrow P$ : a trivialization  $M \times G \rightarrow P$  is given by  $(x, g) \mapsto s(x) \cdot g$ . The reciprocal property is obvious.

**Example 1.1.2** (Hopf fibration). The *Hopf fibration*  $\pi : S^3 \rightarrow S^2$  is a principal bundle with structural group  $S^1$ . It can be defined by considering  $S^3$  and  $S^1$  as the standard unit spheres in  $\mathbb{C}^2$  and  $\mathbb{C}$ , giving rise to an action of  $S^1$  on  $S^2$  by scalar multiplication. Then  $\pi : S^3 \rightarrow S^2$  can be considered as the projection to the orbit space,  $S^3 \rightarrow S^3/S^1 \equiv \mathbb{CP}^1 \approx S^2$  [4, pp. 244]. This principal bundle is nontrivial because the fundamental groups of the total spaces are different. With more generality, for any free right action of a compact Lie group  $G$  on a manifold  $P$ , by the slice theorem, the orbit space  $P/G$  is a manifold and the canonical projection  $P \rightarrow P/G$  is a principal  $G$ -bundle.

**Example 1.1.3** (Frame bundle). Let  $\eta : E \rightarrow M$  be a  $C^\infty$  vector bundle of rank  $r$ . A *frame*  $v$  at a point  $x \in X$  is an ordered basis for the vector space  $E_x$ . In other words, it is a linear isomorphism  $e : \mathbb{R}^r \rightarrow E_x$ . The set  $F_x$  of all frames at  $x$  has a natural right action by the general linear group  $\text{GL}(r, \mathbb{R})$ : every  $g \in \text{GL}(r, \mathbb{R})$  acts on the frame  $e$  via composition to give a new frame,  $e \circ g : \mathbb{R}^r \rightarrow E_x$ . This action is free and transitive on  $F_x$ , since there is a unique invertible linear transformation sending one basis onto another in  $\mathbb{R}^r$ . Then

$$F(E) = \bigsqcup_{x \in M} F_x$$

becomes a  $\text{GL}(r, \mathbb{R})$ -principal bundle over  $M$ , so that the fiber over every  $x \in M$  is  $F_x$ , and the local trivializations canonically induced by the local trivializations of  $E$ . This structure is known as the *frame bundle* of the vector bundle  $E$  [4, pp. 246-247].

*Remark 1.1.4.* Like in Example 1.1.3, on Euclidean (respectively, Hermitian) vector bun-

dles, one can similarly define the principal bundle of orthonormal frames with structural group  $O(r)$  (respectively,  $U(r)$ ). If the bundle is also orientable, the  $SO(r)$ -principal bundle of positively oriented orthonormal frames can also be built. The latter will be a central piece of this discussion.

**Example 1.1.5** (Vector bundle associated to a representation). Let  $P$  be a principal bundle with structural group  $G$  and let  $\rho : G \rightarrow GL(V)$  be a representation of  $G$  on a vector space  $V$ . Define the following right  $G$ -action on  $P \times V$ :

$$(e, v)g = (eg, \rho(g^{-1})v).$$

The quotient space  $P \times_{\rho} V$  of  $P \times V$  by this action is a vector bundle over  $M$  with typical fiber  $V$  [6, p. 23]. It is known as the vector bundle *associated* to  $P$  by  $\rho$ . The element of  $P \times_{\rho} V$  represented by every  $(e, v) \in E \times V$  is denoted by  $[e, v]$ .

## 1.2 Connections

The idea behind connections is extending the notion of the Euclidean directional derivative to a more general definition. This can be achieved for a wide array of mathematical structures. This work will keep its focus mostly on vector bundles, since it is the most useful case for the task at hand. However, some understanding of connections on principal bundles will be also needed, which will be given using the intuitive idea of parallel transport.

### 1.2.1 Connections on vector bundles

Let  $\pi : E \rightarrow M$  be a vector bundle over a manifold  $M$ . A *connection* on  $E$  is a linear map

$$\nabla : \mathfrak{X}(M) \otimes \Gamma(E) \longrightarrow \Gamma(E)$$

such that the following properties hold for all  $X \in \mathfrak{X}(M)$ ,  $s \in \Gamma(E)$  and  $f \in \mathcal{F} := C^{\infty}(M)$ :

- (i)  $\nabla_{fX}s = f\nabla_Xs$ .
- (ii)  $\nabla_X(fs) = (Xf)s + f\nabla_Xs$  (Leibniz rule).

*Remark 1.2.1.* By (i) and (ii), for any  $p \in M$ , the value of  $(\nabla_Xs)(p)$  only depends on the value of  $X$  at  $p$  and the values of  $s$  along any given smooth path  $c : (-\epsilon, \epsilon) \rightarrow M$  with  $c(0) = p$  and  $c'(0) = X(p)$ . It follows that  $\nabla$  can also be regarded as a linear map  $\nabla : \Gamma(E) \rightarrow \Gamma(T^*M \otimes E)$  (the space of  $E$ -valued 1-forms) satisfying  $\nabla(fs) = (df) \cdot s + f\nabla s$ . This alternative definition is very useful in some cases.

A connection on a Euclidean/Hermitian vector bundles  $E$  over  $M$  is said to be *compatible with the metric* or *metric* if, for every  $X \in \mathfrak{X}(M)$  and  $s, t \in \Gamma(E)$ ,

$$Xg(s, t) = g(\nabla_X s, t) + g(s, \nabla_X t).$$

Any Euclidean/Hermitian vector bundle admits a metric connection [4, pp. 73-76].

The concept of connection leads to several geometric ideas, the most important of which is the curvature operator. The *curvature operator*  $K$  of a connection  $\nabla$  on a vector bundle  $E$  is a linear map

$$K : \mathfrak{X}(M) \otimes \mathfrak{X}(M) \otimes \Gamma(E) \longrightarrow \Gamma(E)$$

defined by

$$K(X, Y)s \equiv K_{X,Y}s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]}s.$$

Since  $K$  is  $\mathcal{F}$ -linear (in all three arguments), it is defined pointwise. If  $K = 0$ , then  $\nabla$  is said to be *flat*.

A connection on the tangent bundle  $\pi : TM \rightarrow M$  of a manifold  $M$  is more commonly referred to as an *affine connection*. The *torsion* of an affine connection  $\nabla$  is the linear map

$$T : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$$

defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

This operator is also  $\mathcal{F}$ -linear (in its two arguments), and therefore it is defined pointwise. If  $T = 0$ , then  $\nabla$  is said to be *symmetric* or *torsion-free*.

### 1.2.2 Parallel transport and connections on principal bundles

Let  $\pi : E \rightarrow M$  be a vector bundle equipped with a connection  $\nabla$ . For every piecewise smooth path  $c : [a, b] \rightarrow M$  and  $v \in E_{c(a)}$ , there is a unique piecewise smooth path  $\tilde{c}_v : [a, b] \rightarrow E$  such that  $\tilde{c}_v(a) = v$ ,  $\pi \circ \tilde{c} = c$  ( $\tilde{c}$  is a lift of  $c$ ) and  $\nabla_{\tilde{c}'} \tilde{c}_v = 0$ . Moreover, for each  $t \in [a, b]$ , the mapping  $v \mapsto \tilde{c}_v(t)$  defines a linear isomorphism  $\mathbb{P}_{c,t} : E_{c(a)} \rightarrow E_{c(t)}$ , called the *parallel transport* along  $c$  up to  $t$ . This parallel transport  $\mathbb{P}$  determines  $\nabla$ :

$$\nabla_{c'(a)} s = \left. \frac{d}{dt} \mathbb{P}_{c,t}^{-1} s c(t) \right|_{t=a} \in E_{c(a)}.$$

Thus a connection can be also described by giving an abstract parallel transport satisfying certain properties.

Going one step further, the connection on  $E$  induces a so-called *connection* in the principal frame bundle  $F(E)$  (Example 1.1.3), which can be defined by extending the idea of parallel

transport. For any piecewise smooth path  $c : [a, b] \rightarrow M$ ,  $t \in [a, b]$  and  $e : \mathbb{R}^r \rightarrow E_{c(a)}$  in  $F_{c(a)}$ , let  $\widehat{\mathbb{P}}_{c,t}(e)$  be the composition

$$\mathbb{R}^r \xrightarrow{e} E_{c(a)} \xrightarrow{\mathbb{P}_{c,t}} E_{c(t)}.$$

This defines a  $\mathrm{GL}(r, \mathbb{R})$ -equivariant diffeomorphism  $\widehat{\mathbb{P}}_{c,t} : F_{c(a)} \rightarrow F_{c(t)}$ , called the *parallel transport* in  $F(E)$  along  $c$  up to  $t$ .

The above notion can be extended to define a *parallel transport*  $\widehat{\mathbb{P}}$  on a principal  $G$ -bundle  $P$  over  $M$ , assigning a  $G$ -equivariant diffeomorphism  $\widehat{\mathbb{P}}_{c,t} : P_{c(a)} \rightarrow P_{c(t)}$  to any piecewise smooth path  $c : [a, b] \rightarrow M$  and  $t \in [a, b]$ , satisfying certain properties. Such a parallel transport is a way to describe connections on principal bundles.

Now, take a representation  $\rho : G \rightarrow \mathrm{GL}(V)$  and consider the associated vector bundle  $P \times_{\rho} V$  (Example 1.1.5). Then  $\widehat{\mathbb{P}}$  induces a connection on  $P \times_{\rho} V$ , whose parallel transport  $\mathbb{P}$  is given by  $\mathbb{P}_{c,t}([e, v]) = [\widehat{\mathbb{P}}_{c,t}(e), v]$ .

A more detailed description of connections on principal bundles is far from the main goals of this survey, so it is omitted. There is a large amount of literature regarding connections on principal bundles and associated vector bundles. The reader is referred to [4, Section 28], [17, Chapter 2].

### 1.3 Fundamentals of Riemannian geometry

The most basic notion that this work pulls from differential geometry is that of a Riemannian manifold. A *Riemannian metric*  $g$  on a manifold  $M$  is a Euclidean structure on  $TM$ . Every manifold admits a Riemannian metric [4, p. 6]. An *isometry* between Riemannian manifolds is a diffeomorphism whose tangent map preserves the Riemannian metric at every tangent space. *Local isometries* are similarly defined using local diffeomorphisms.

A particular, and greatly important, case of affine connection is the *Levi-Civita connection* of a Riemannian manifold  $M$ , which is the unique symmetric metric connection on  $M$  [15, p. 16-17], [6, p. 12]. Henceforth,  $M$  will be equipped with the Levi-Civita connection  $\nabla$ . The curvature operator of the Levi-Civita connection is often referred to as the *Riemann curvature operator*, and denoted by  $R$ . In this case, in terms of a local frame  $(e_1, \dots, e_n)$  of  $TM$  corresponding to local coordinates  $x^i$ , the curvature operator can be put in a more physics-friendly, tensor form:

$$R_{e_j, e_k} e_l = R_{ljk}^i e_i,$$

where Einstein notation has been used. The Riemann curvature operator satisfies, for all  $X, Y, W, Z \in \mathfrak{X}(M)$ ,

$$\begin{aligned} R_{X,Y}Z + R_{Y,Z}X + R_{Z,X}Y &= 0 \quad (\text{the first Bianchi identity}), \\ g(R_{X,Y}Z, W) + g(R_{X,Y}W, Z) &= 0. \end{aligned} \quad (1.3.1)$$

The symmetries of  $R$  leave only one nontrivial contraction of the indices [15, p. 58-60], [6, p. 14]. This is known as the *Ricci curvature tensor*, and can be expressed, in components, as  $\text{Ric}_{ab} = R_{aib}^i$ . It is symmetric.

The last possible contraction is the *scalar curvature*,

$$S = g^{ab}\text{Ric}_{ab}.$$

**Example 1.3.1** (Flat Euclidean metric). The simplest and most well-known example of a Riemannian manifold is  $\mathbb{R}^n \equiv (\mathbb{R}^n, g^c)$ , where, using the standard coordinates on  $\mathbb{R}^n$ , the metric  $g^c$  is given by

$$g_p^c((\partial_{x^i})_p, (\partial_{x^j})_p) = \delta_{ij}.$$

This is often called the *standard Riemannian structure* on  $\mathbb{R}^n$ , or the *canonical Euclidean metric*. This Riemannian manifold is flat. Reciprocally, any flat Riemannian manifold is locally isometric to  $\mathbb{R}^n$ .

Obviously, much more intricate Riemannian manifolds can be built.

## 1.4 Differential operators

The theory of differential operators on manifolds provides most of the analytical tools one needs to study Dirac Operators.

*Notation 1.4.1.* A *multi-index* is an  $n$ -tuple  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ . Set  $|\alpha| = \sum_{k=1}^n \alpha_k$ . For each  $\xi \in \mathbb{C}^n$ , let  $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$ . Multi-index notation is used for differentiation with respect to local coordinates  $x^i$  on a manifold  $M$ :

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} \equiv \frac{\partial^{|\alpha|}}{\partial (x^1)^{\alpha_1} \dots \partial (x^n)^{\alpha_n}} \equiv \frac{\partial^{|\alpha|}}{\partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}}.$$

Let  $E$  and  $F$  be two smooth complex vector bundles over  $M$  of ranks  $r$  and  $q$ , respectively. A (*linear*) *differential operator* of order  $k$  on  $M$  is a linear map  $P : \Gamma(E) \rightarrow \Gamma(F)$  satisfying the following property. For each point  $p \in M$ , take a neighborhood  $U$  with local coordinates  $x^i$ , and local trivializations of  $E$  and  $F$  on  $U$ , giving rise to identities  $\Gamma_c(U, E) \equiv C_c^\infty(U, \mathbb{C}^r)$

and  $\Gamma_c(U, F) \equiv C_c^\infty(U, \mathbb{C}^q)$ . Then  $P$  is required to satisfy  $P(\Gamma_c(U, E)) \subset \Gamma_c(U, F)$  ( $P$  is *local*), and the restriction  $P : \Gamma_c(U, E) \equiv C_c^\infty(U, \mathbb{C}^r) \rightarrow \Gamma_c(U, F) \equiv C_c^\infty(U, \mathbb{C}^q)$  must be of the form

$$P = \sum_{|\alpha| \leq k} A^\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha},$$

where the local coefficients  $A^\alpha$  are smooth maps  $U \rightarrow M_{r \times q}(\mathbb{C}) \equiv \mathbb{C}^{rq}$  with  $A^\alpha \neq 0$  for some  $\alpha$  with  $|\alpha| = k$ .

**Example 1.4.2** (Connections). A connection  $\nabla : \Gamma(E) \rightarrow \Gamma(T^*M \otimes E)$  on a vector bundle  $E$  over  $M$  is a differential operator of order one. In fact, it is *universal* in the sense that any differential operator of order one  $P : \Gamma(E) \rightarrow \Gamma(F)$  on  $M$  is a composition

$$\Gamma(E) \xrightarrow{\nabla} \Gamma(T^*M \otimes E) \xrightarrow{A} \Gamma(F),$$

for some zero order differential operator  $A$ .

For a differential operator  $P : \Gamma(E) \rightarrow \Gamma(F)$  of order  $k$  on  $M$ , its *principal symbol*  $\sigma(P)$  is a map that associates each point  $x \in M$  and each covector  $\xi = \xi_j dx^j \in T_x^*M$  (using Einstein notation), to a linear map

$$\sigma_\xi(P) = i^k \sum_{|\alpha|=k} A^\alpha(x) \xi^\alpha : E_x \longrightarrow F_x.$$

There is a coordinate-free way to define  $\sigma(P)$  as a section of  $(\odot^k TM) \otimes \text{Hom}(E, F)$  [8, p. 168]. It shows that the principal symbol does not depend on a particular choice of local coordinates or trivializations of the bundles. However, for the purposes of this work, the above definition of  $\sigma(P)$  will suffice as a means to analyze the Dirac operators.

The invariance of the principal symbol under changes in local trivializations and coordinates allows for different classifications of differential operators, regarding what properties it possesses for certain classes of operators. The relevant class for this work is the following. It is said that  $P$  is *elliptic* if, for every nonzero covector  $\xi \in T_x^*M$ , the principal symbol  $\sigma_\xi(P) : E_x \rightarrow F_x$  is an invertible mapping.

Real differential operators and their symbols are defined in the same manner, only replacing  $\mathbb{C}$  with  $\mathbb{R}$ .

**Example 1.4.3** (Laplace-Beltrami operator). Let  $(M^n, g)$  be a Riemannian manifold. Consider the map  $\Delta : C^\infty(M) \rightarrow C^\infty(M)$  on the space of smooth functions on  $M$  given, in local coordinates  $x^i$ , by

$$\Delta = \frac{1}{\sqrt{g}} \partial_j \left( \sqrt{g} g^{jk} \partial_k \right) = g^{jk} \frac{\partial^2}{\partial_j \partial_k} + \text{lower order terms},$$

where Einstein notation has been used. Here,  $g_{jk} dx^j dx^k$  is the metric tensor,  $g = \det(g_{jk})$  and  $(g^{jk}) = (g_{jk})^{-1}$ . This mapping is known as the *Laplace-Beltrami operator*, and it is a second order differential operator. The principal symbol is easily determined for any cotangent vector  $\xi = \xi_k dx^k$ :

$$\sigma_\xi(P) = -g^{jk} \xi_j \xi_k = -|\xi|^2,$$

which is invertible (as a complex linear map) for each  $\xi \neq 0$ . The Laplace-Beltrami operator is hence elliptic.

The following is a fundamental result for elliptic operators, which will be essential for the study of the Dirac operator in particular:

**Theorem 1.4.4.** *Let  $M$  be an  $n$ -dimensional compact Riemannian manifold, and let  $P : \Gamma(E) \rightarrow \Gamma(E)$  be a self-adjoint elliptic differential operator of order  $k > 0$ . Then, the spectrum of  $P$  is real and discrete. Each eigenspace of  $P$  is finite-dimensional and consists of smooth sections. Moreover, the eigenspaces  $E_\lambda$  ( $\lambda \in \text{Spec}(P)$ ) of  $P$  provide an orthogonal direct sum decomposition for the Hilbert space  $L^2(E)$ ,*

$$L^2(E) = \bigoplus_{\lambda} E_{\lambda}.$$

*Proof.* It is somewhat technical and does not provide much insight on the topic of this survey. See [8, pp. 196-197]. ■





## Chapter 2

# Spin manifolds and the spinor bundle

The first step in the journey to build the Dirac operator is defining the structure on which it operates. As previously stated, this endeavor not only requires pulling tools from differential geometry, but also some algebraic notions. This chapter is devoted to exploring those notions and intertwining them with previous geometric aspects to further the reader's understanding of spin geometry.

### 2.1 Spin groups and their representations

This section presents the essential algebraic ideas at the heart of spin geometry. They provide the necessary tools to build the *spinor bundle*, which is essential in this discussion, since the Dirac operator acts on the space of its smooth sections.

#### 2.1.1 Clifford algebras

**Definition 2.1.1.** Let  $V$  be a vector space over a commutative field  $\mathbb{K}$ . Let

$$\mathcal{T}(V) = \bigoplus_{r=0}^{\infty} \bigotimes_r V$$

denote the tensor algebra of  $V$  (see [7] for more information on tensor algebras). Suppose  $q$  is a quadratic form on  $V$ , and define  $\mathcal{J}_q(V)$  to be the ideal in  $\mathcal{T}(V)$  generated by the elements of the form  $v \otimes v + q(v)1$  (where  $1$  is the unit of  $\mathcal{T}(V)$ ). The *Clifford algebra* of

the quadratic vector space  $(V, q)$  is the quotient

$$\text{Cl}(V, q) = \mathcal{T}(V) / \mathcal{J}_q(V).$$

The product in  $\text{Cl}(V, q)$  (*Clifford product*) shall be denoted by “ $\cdot$ ”. When there is no confusion, this symbol can be dropped to simplify the notation.

*Remark 2.1.2.* There is a natural embedding  $j_q : V \hookrightarrow \text{Cl}(V, q)$  given by the composition

$$V \xrightarrow{i} \mathcal{T}(V) \xrightarrow{\pi_q} \text{Cl}(V, q),$$

where  $i$  is the natural inclusion and  $\pi_q$  is the canonical projection map (see [8, p. 8] for the proof that this is an embedding). Thus  $V$  can be considered as a subset of  $\text{Cl}(V, q)$ . Objects of the form  $j_q(v)$  ( $v \in V$ ) will be referred to as  $v \in \text{Cl}(V, q)$ .

$\text{Cl}(V, q)$  can be considered as the associative algebra with unit, generated by elements of  $V$ , and subject to the relations  $v^2 = -q(v)1$  (when no confusion is likely to occur, the unit symbol may be dropped). Hence for all  $v, w \in V$ ,

$$v \cdot w + w \cdot v = -2q(v, w), \quad (2.1.1)$$

where  $2q(v, w) = q(v+w) - q(v) - q(w)$  is the polarization<sup>1</sup> of  $q$ . If  $\mathbb{K}$  is not of characteristic 2, then the relations (2.1.1) are enough to describe  $\text{Cl}(V, q)$ , which is not the case if  $\mathbb{K}$  is of characteristic 2.

These relations provide a very useful universal characterization of Clifford algebras.

**Proposition 2.1.3** (Universal property). *Let  $\mathcal{A}$  be an associative  $\mathbb{K}$ -algebra with unit, and let  $j : V \rightarrow \mathcal{A}$  be a linear map satisfying  $j(v)^2 = -q(v)1_{\mathcal{A}}$ , for every  $v \in V$ . Then  $j$  extends uniquely to a  $\mathbb{K}$ -algebra homomorphism  $\tilde{j} : \text{Cl}(V, q) \rightarrow \mathcal{A}$ . Moreover,  $\text{Cl}(V, q)$  is the only associative  $\mathbb{K}$ -algebra (up to isomorphism) that satisfies this property.*

*Proof.* (Existence of the homomorphism.) For any linear map  $f : V \rightarrow \mathcal{A}$ , there is a unique  $\mathbb{K}$ -algebra homomorphism  $\bar{f} : \mathcal{T}(V) \rightarrow \mathcal{A}$  so that the diagram

$$\begin{array}{ccc} \mathcal{T}(V) & \xrightarrow{\bar{f}} & \mathcal{A} \\ \uparrow i & \nearrow f & \\ V & & \end{array}$$

---

<sup>1</sup>Writing  $q(v, w)$  is an abuse of notation, since  $q$  is a quadratic form. For the sake of simplicity,  $q$  and its associated symmetric bilinear form will both be referred to as  $q$ , since there is no confusion as to which one is being used in each context.

commutes [7]. In particular, the relation  $j(v)^2 = -q(v)1_{\mathcal{A}}$  guarantees that  $\bar{j} = 0$  on  $\mathcal{J}_q$ , so  $\bar{j}$  descends to  $\text{Cl}(V, q)$  and  $\tilde{j} = \bar{j} \circ \pi_q$ .

(Uniqueness of  $\text{Cl}(V, q)$ .) Suppose that  $\mathcal{C}$  is an associative  $\mathbb{K}$ -algebra with unit and that  $i : V \rightarrow \mathcal{C}$  is an embedding such that any  $j$  that satisfies the conditions in the proposition can be uniquely extended to a  $\mathbb{K}$ -algebra homomorphism  $\tilde{j} : \mathcal{C} \rightarrow \mathcal{A}$ . Then the isomorphism from  $V \subset \text{Cl}(V, q)$  to  $i(V) \subset \mathcal{C}$  induces an algebra isomorphism  $\text{Cl}(V, q) \xrightarrow{\cong} \mathcal{C}$ . ■

This characterization is highly useful when discussing other properties of Clifford algebras.

**Proposition 2.1.4.** *Every Clifford algebra  $\text{Cl}(V, q)$  has a unique canonical antiautomorphism  $t$ , called the “transpose,” which satisfies, for all  $v \in V$ ,*

$$t^2 = \text{id}, \quad t(v) = v.$$

*Proof.* Consider the algebra  $\text{Cl}(V, q)^o$ , given by  $\text{Cl}(V, q)$  with the product  $*$  defined by  $x * y = y \cdot x$  for all  $x, y \in \text{Cl}(V, q)$ . Since it satisfies the universal property, there is a unique isomorphism  $\text{Cl}(V, q) \xrightarrow{\bar{t}} \text{Cl}(V, q)^o$ . It follows that, for  $x \in \text{Cl}(V, q)$ ,  $t(x) = \bar{t}(x) \in \text{Cl}(V, q)$ . The properties of  $t$  follow trivially from its definition. ■

The transpose  $t(x)$  of an element  $x \in \text{Cl}(V, q)$  can also be written as  $x^t$ .

**Proposition 2.1.5.** *Every Clifford algebra  $\text{Cl}(V, q)$  has a unique canonical automorphism  $\alpha$  which satisfies, for all  $v \in V$ ,*

$$\alpha^2 = \text{id}, \quad \alpha(v) = -v.$$

*Proof.* Consider the linear map  $\alpha_0 : V \rightarrow \text{Cl}(V, q)$ , defined by  $\alpha_0(v) = -v$ . Apply the universal property to find an automorphism  $\alpha$  that satisfies the desired properties. ■

The automorphism  $\alpha$  yields a useful decomposition of  $\text{Cl}(V, q)$ :

$$\text{Cl}(V, q) = \text{Cl}^0(V, q) \oplus \text{Cl}^1(V, q),$$

where  $\text{Cl}^i(V, q) = \{x \in \text{Cl}(V, q) : \alpha(x) = (-1)^i x\}$  are the eigenspaces of  $\alpha$ .  $\text{Cl}^0(V, q)$  is the *even part* of  $\text{Cl}(V, q)$ , and  $\text{Cl}^1(V, q)$  is the *odd part*. Given that  $\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$ , and taking indexes  $i, j$  modulo 2, it is clear that

$$\text{Cl}^i(V, q) \cdot \text{Cl}^j(V, q) \subset \text{Cl}^{i+j}(V, q).$$

This makes  $\text{Cl}(V, q)$  into a  $\mathbb{Z}_2$ -graded algebra. Note that  $\text{Cl}^0(V, q)$  is a subalgebra, but  $\text{Cl}^1(V, q)$  is not.

As a last annotation on the topic of Clifford algebras, they present an interesting relationship with the *exterior algebra*  $\bigwedge^\bullet V$  (also called the “Grassmann algebra”), which is the Clifford algebra of  $V$  defined by the quadratic form  $q \equiv 0$ ; that is,  $\bigwedge^\bullet V = \text{Cl}(V, 0)$ .

**Proposition 2.1.6.** *Every Clifford algebra  $\text{Cl}(V, q)$  is isomorphic, as a vector space, to the exterior algebra  $\bigwedge^\bullet V$ . This isomorphism is natural when  $\mathbb{K}$  is not of characteristic 2.*

*Proof.* It is not particularly informative for the purposes of this work (see [9]). In the finite-dimensional case, it is given by the mapping  $e_{i_1} \wedge \cdots \wedge e_{i_k} \mapsto e_{i_1} \cdots e_{i_k}$  of basis elements. ■

*Remark 2.1.7.* The proposition means that if  $V$  is a vector space of dimension  $n \in \mathbb{N}$ , then  $\dim(\text{Cl}(V, q)) = 2^n$ . Note that  $\text{Cl}(V, q)$  and  $\bigwedge^\bullet V$  are not isomorphic as algebras, except when  $q \equiv 0$ .

### 2.1.2 The Pin and Spin groups

So far, this work has not gone into specific notions of spin geometry. This will change in this section. Consider the multiplicative group of units in the Clifford algebra:

$$\text{Cl}^*(V, q) = \{x \in \text{Cl}(V, q) : \exists x^{-1}, x^{-1}x = xx^{-1} = 1\}.$$

This group contains all elements  $v \in V$  such that  $q(v) \neq 0$ . Since  $v^2 = -q(v)1$ , their inverses are

$$v^{-1} = -\frac{v}{q(v)}. \quad (2.1.2)$$

The group  $P(V, q) \subset \text{Cl}^*(V, q)$  generated by the elements  $v \in V$  such that  $q(v) \neq 0$  contains some very important subgroups which lie at the heart of spin geometry.

**Definition 2.1.8.** The *Pin group* of the quadratic vector space  $(V, q)$  is the subgroup  $\text{Pin}(V, q) \subset P(V, q)$  generated by the elements  $v \in V$  with  $q(v) = \pm 1$ . The associated *Spin group* is just its even part,

$$\text{Spin}(V, q) = \text{Pin}(V, q) \cap \text{Cl}^0(V, q).$$

Analysis of the most crucial properties of these groups entails some somewhat cumbersome, but inevitable, preliminary legwork. If the reader is not interested in technical fiddles, and wants to get on with topics more directly related to Dirac operators, Theorems 2.1.15 and

2.1.16 represent the culmination of the work in this section, and provide the necessary information to advance to the next one.

The first step is introducing the *twisted adjoint representation* [10], a homomorphism

$$\rho: \text{Cl}^*(V, q) \longrightarrow \text{GL}(\text{Cl}(V, q)), \quad \rho_x(y) = \alpha(x)yx^{-1}.$$

This representation arises as a means to bypass the shortcomings of the usual adjoint representation. Any of the specialized references in this section provide an explanation as to why the twisted version is necessary.

From now on, assume that  $\mathbb{K}$  is not of characteristic 2. This is necessary for the following line of reasoning.

**Proposition 2.1.9.** *Let  $v \in V \subset \text{Cl}(V, q)$  be an element with  $q(v) \neq 0$ . Then  $\rho_v(V) = V$ , and for all  $w \in V$ , the following equation holds:*

$$\rho_v(w) = w - 2\frac{q(v, w)}{q(v)}v.$$

*Proof.* From (2.1.2), it follows that  $q(v)v^{-1} = -v = \alpha(v)$ . Thus

$$q(v)\rho_v(w) = q(v)\alpha(v)wv^{-1} = \alpha(v)w\alpha(v) = v w v. \quad (2.1.3)$$

Now, applying (2.1.1) to (2.1.3),

$$q(v)\rho_v(w) = -v^2w - 2q(v, w)v = q(v)w - 2q(v, w)v.$$

From this equation, it is clear that  $\rho_v(V) = V$ , since  $\rho_v(v) = -v$  and  $\rho_v$  fixes any  $w \in v^\perp$ , thus completing the proof. ■

Proposition 2.1.9 shows that  $\rho_v(w)$  is a reflection of  $w \in V$  across  $v^\perp$ . This characterization will be important later.

Using the twisted adjoint representation, the following subgroup can be defined:

$$\tilde{\text{P}}(V, q) = \{x \in \text{Cl}^*(V, q) : \rho_x(V) = V\}.$$

The *norm mapping*  $N: \text{Cl}(V, q) \rightarrow \text{Cl}(V, q)$  will also be needed:

$$N(x) := x \cdot \alpha(x^t). \quad (2.1.4)$$

**Proposition 2.1.10.** *Let  $V$  be finite-dimensional and let  $q$  be nondegenerate (i.e. there are no zeroes in its signature). Then the kernel of the homomorphism given by the restriction of  $\rho$  to  $\tilde{\text{P}}(V, q)$ ,*

$$\rho: \tilde{\text{P}}(V, q) \longrightarrow \text{GL}(V),$$

*is the group  $\mathbb{K}^*$  of nonzero multiples of the identity in  $\tilde{\text{P}}(V, q)$ .*

*Proof.* It is not overly interesting for this analysis. See [8, pp. 14-15]. ■

**Proposition 2.1.11.** *Let  $V$  be finite-dimensional and let  $q$  be nondegenerate. Then restricting  $N$  to the group  $\tilde{P}(V, q)$  yields a homomorphism*

$$N : \tilde{P}(V, q) \longrightarrow \mathbb{K}^*$$

*into the multiplicative group of nonzero multiples of the identity in  $\text{Cl}(V, q)$ .*

*Proof.* First, it shall be proved that  $N$  is well-defined. To that end, note that  $\rho_x(v) \in V$ , for every  $x \in \tilde{P}(V, q)$ , and  $v \in V$ . Now, apply  $t$  to find

$$t(\rho_x(v)) = (x^{-1})^t v \alpha(x^t) = \alpha(x) v x^{-1} = \rho_x(v),$$

since  $t(w) = w$  for all  $w \in V$ . Hence

$$x^t \alpha(x) v x^{-1} (\alpha(x^t))^{-1} = \alpha[\alpha(x^t)x] v [\alpha(x^t)x]^{-1} = \rho_{\alpha(x^t)x}(v) = v.$$

From this equality, it follows that  $\rho_{\alpha(x^t)x} = \text{id}$ . In other words,  $\alpha(x^t)x \in \ker(\rho)$ . By Proposition 2.1.10,  $\alpha(x^t)x \in \mathbb{K}^*$ . Apply  $\alpha$  to show that  $x^t \alpha(x) = N(x^t) \in \mathbb{K}^*$ . On the other hand,  $t$  preserves  $\tilde{P}(V, q)$ , so it is clear that  $N(x) \in \mathbb{K}^*$ , and  $N$  is well-defined.

Now, to prove that it is a homomorphism, take  $x, y \in \tilde{P}(V, q)$ . Then

$$N(xy) = xy \alpha(y^t) \alpha(x^t) = x N(y) \alpha(x^t).$$

Given that  $N(y) \in \mathbb{K}^*$ , it is clear that  $N(xy) = N(x)N(y)$ , making  $N$  a homomorphism on  $\tilde{P}(V, q)$ . ■

Immediate consequences can be inferred from Proposition 2.1.11. To that end, define the *orthogonal group* of  $V$  with respect to  $q$  as

$$\text{O}(V, q) = \{\lambda \in \text{GL}(V) : \lambda^* q = q\},$$

where  $\lambda^* q$  is the pullback of  $q$  by  $\lambda$ , defined by  $\lambda^* q(v) = q(\lambda(v))$ .

**Corollary 2.1.12.** *The mappings  $\rho_x : V \rightarrow V$  for  $x \in \tilde{P}(V, q)$  preserve the quadratic form  $q$ , obtaining a group homomorphism*

$$\rho : \tilde{P}(V, q) \longrightarrow \text{O}(V, q).$$

*Proof.* For  $x \in \tilde{P}(V, q)$ , one easily determines  $N(\alpha(x)) = \alpha(x)x^t = \alpha(N(x)) = N(x)$ . Now, define  $V^\times = \{v \in V : q(v) \neq 0\} \subset \tilde{P}(V, q)$  (the set of generators of  $\tilde{P}(V, q)$ ). Then, for every  $v \in V^\times$ ,

$$N(\rho_x(v)) = N(\alpha(x)vx^{-1}) = N(\alpha(x))N(v)N(x)^{-1} = N(v) = q(v),$$

where the last equality comes from applying  $v^t = v$  and  $v\alpha(v) = q(v)$  to (2.1.4). Since  $\rho_x(v) \in V$ , it is clear that  $\rho_x$  preserves nonzero  $q$ -lengths. Now, apply  $\rho_{x^{-1}}$  to show that  $\rho_x(V^\times) = V^\times$ , so  $\rho_x$  leaves invariant the set of vectors of zero  $q$ -length. It follows that  $\rho_x$  is  $q$ -orthogonal.  $\blacksquare$

With this corollary, all the preliminary pieces of the analysis of the Pin and Spin groups start to fall into place. Only one more result is needed before one can proceed.

**Theorem 2.1.13** (Cartan-Dieudonné, version of the proof in [11]). *Let  $V$  be a finite-dimensional vector space endowed with a nondegenerate quadratic form  $q$ . Then every element  $\lambda \in O(V, q)$  can be written as a product of  $r$  reflections, i.e.*

$$\lambda = \rho_{v_1} \circ \cdots \circ \rho_{v_r},$$

where  $r \leq \dim(V)$ .

The Cartan-Dieudonné Theorem, together with Corollary 2.1.12, provide some essential insight into the properties of the Pin and Spin groups for spaces  $(V, q)$  such that  $V$  is finite-dimensional and  $q$  is nondegenerate. Recall that, by definition,

$$P(V, q) = \{v_1 \cdots v_r \in \text{Cl}(V, q) : v_1, \dots, v_r \in V^\times\} \subset \tilde{P}(V, q),$$

so the restriction of  $\rho$  to  $P(V, q)$  is a homomorphism with a rather simple definition. For  $x = v_1 \cdots v_r \in P(V, q)$ ,

$$\rho_x = \rho_{v_1} \circ \cdots \circ \rho_{v_r},$$

where, by Proposition 2.1.9,  $\rho_{v_i}$  is the reflection across  $v_i^\perp$ . Thus one finds (with a minor abuse of notation) the homomorphism  $\rho : P(V, q) \rightarrow O(V, q)$ . The Cartan-Dieudonné Theorem guarantees that this homomorphism is surjective. Moreover, defining  $\text{SP}(V, q) = P(V, q) \cap \text{Cl}^0(V, q)$  and considering the special orthogonal group

$$\text{SO}(V, q) = \{\lambda \in O(V, q) : \det(\lambda) = 1\} = \{\rho_{v_1} \circ \cdots \circ \rho_{v_r} \in O(V, q) : r \text{ is even}\},^2$$

one obtains a new homomorphism  $\text{SP}(V, q) \xrightarrow{\rho} \text{SO}(V, q)$ .

The end of this analysis is already in sight. The objective now is proving that the homomorphism  $\rho$ , when restricted to the groups

$$\text{Pin}(V, q) = \{v_1 \cdots v_r \in P(V, q) : q(v_j) = \pm 1 \ \forall j = 1, \dots, r\},$$

$$\text{Spin}(V, q) = \{v_1 \cdots v_r \in \text{Pin}(V, q) : r \text{ is even}\},$$

is still surjective. The answer, as it turns out, is affirmative when the underlying field  $\mathbb{K}$  has the property discussed below.

---

<sup>2</sup>This equality comes from the fact that the determinant of a reflection is always -1.

**Definition 2.1.14.** Let  $\mathbb{K}$  be a field of characteristic  $\neq 2$ . Then  $\mathbb{K}$  is *spin* if at least one of the equations  $t^2 = \pm a$  can be solved in  $\mathbb{K}$  for each  $a \in \mathbb{K}^*$ .

Most of the more well-known fields are spin. For example,  $\mathbb{R}$  and  $\mathbb{C}$  both verify the spin condition. The real case holds most of the practical interest for the purposes of this work.

Finally, all the information contained in this section comes together for its main result.

**Theorem 2.1.15.** *Let  $V$  be a finite-dimensional vector space over a spin field  $\mathbb{K}$ . Let  $q$  be a nondegenerate quadratic form on  $V$ , and  $i = \sqrt{-1}$ . There are short exact sequences*

$$\begin{aligned} 1 \rightarrow F \rightarrow \text{Pin}(V, q) &\xrightarrow{\rho} \text{O}(V, q) \rightarrow 1, \\ 1 \rightarrow F \rightarrow \text{Spin}(V, q) &\xrightarrow{\rho} \text{SO}(V, q) \rightarrow 1, \end{aligned}$$

where

$$F = \begin{cases} \mathbb{Z}_2 = \{1, -1\} & \text{if } i \notin \mathbb{K} \\ \mathbb{Z}_4 = \{\pm 1, \pm i\} & \text{if } i \in \mathbb{K}. \end{cases}$$

*Proof.* There are two steps in this proof. First, one has to characterize the kernel of  $\rho$  and make sure that it is the image of the monomorphisms  $F \rightarrow \text{Pin}(V, q)$  or  $F \rightarrow \text{Spin}(V, q)$  that canonically maps elements of  $F$  to their analogues in  $\mathbb{K}$  (considered as a subset of the  $\text{Pin}$  or  $\text{Spin}$  groups). Then one must check the surjectivity of the mappings  $\rho$ .

For the first stage, consider an element  $x = v_1 \cdots v_r \in \text{Pin}(V, q)$  such that  $x \in \ker(\rho)$ . By Proposition 2.1.10,  $x \in \mathbb{K}^*$ . Moreover,  $x^2 = N(v_1) \cdots N(v_r) = \pm 1$ . This characterizes  $\ker(\rho)$  for both possible field types.

For the surjectivity of  $\rho$ , consider the fact that the reflection  $\rho_v$  is the same as  $\rho_{tv}$  for every nonzero  $t \in \mathbb{K}$ . Now, take the  $q$ -normalization  $q(tv) = t^2 q(v) = \pm 1$ . Since  $\mathbb{K}$  is spin, at least one of the equations  $t^2 = \pm q(v)^{-1}$  can be solved in  $\mathbb{K}$ . Coupling this fact with the Cartan-Dieudonné Theorem, it is clear that elements of  $\text{O}(V, q)$  or  $\text{SO}(V, q)$  can be written as suitable compositions of  $r$  reflections across the orthogonal subspaces of vectors of  $q$ -length 1. The surjectivity of  $\rho$  immediately follows.  $\blacksquare$

Theorem 2.1.15 is the most important result this work has dealt with so far. It characterizes the Spin group in a way that will be essential in Section 2.1.3. Up until this point, the line of reasoning has been purposefully kept as general as possible, but now it becomes necessary to focus on the real case,  $\mathbb{K} = \mathbb{R}$ .



Let  $V$  be an  $n$ -dimensional real vector space endowed with a positive definite quadratic form  $q$ , so there is a basis for  $V \cong \mathbb{R}^n$  such that

$$q(x) = x_1^2 + \cdots + x_n^2.$$

For the sake of simplicity, the notation will be simplified to  $O(V, q) \equiv O_n$ ,  $SO(V, q) \equiv SO_n$ . Similarly, the corresponding Clifford algebra and the Pin and Spin groups will be named  $Cl_n$ ,  $Pin_n$ ,  $Spin_n$  respectively. In particular,  $Spin_n$  possesses some interesting properties.

**Theorem 2.1.16.** *Let  $n \in \mathbb{N}$ . The group  $Spin_n$  satisfies*

- (i) *For  $n \geq 2$ ,  $Spin_n$  is a connected group.*
- (ii) *For  $n \geq 3$ ,  $Spin_n$  is simply connected and  $\rho : Spin_n \rightarrow SO_n$  is the universal covering of  $SO_n$ .*

*Proof.* Consider the short exact sequence

$$1 \rightarrow \mathbb{Z}_2 \rightarrow Spin_n \xrightarrow{\rho} SO_n \rightarrow 1$$

given by Theorem 2.1.15. This establishes  $Spin_n$  as a twofold cover of  $SO_n$  (since  $\rho_x = \rho_{-x}$ ). To prove that  $Spin_n$  is connected, it is enough [12, p. 16] to find a path connecting the elements  $\pm 1 \in Spin_n$ . To that end, choose two  $q$ -orthogonal vectors  $e_1, e_2 \in \mathbb{R}^n$  such that  $q(e_1) = q(e_2) = 1$ , and define a path  $\gamma : [0, 1] \rightarrow Spin_n$  by

$$\begin{aligned} \gamma(t) &= \cos(\pi t) + \sin(\pi t)e_1e_2 \\ &= \left[ \cos\left(\frac{\pi}{2}t\right)e_1 + \sin\left(\frac{\pi}{2}t\right)e_2 \right] \left[ \sin\left(\frac{\pi}{2}t\right)e_2 - \cos\left(\frac{\pi}{2}t\right)e_1 \right], \end{aligned}$$

which satisfies  $\gamma(0) = 1$ ,  $\gamma(1) = -1$ . The simple connectedness of  $Spin_n$ ,  $n \geq 3$  follows from standard computations in homotopy theory. ■

Basically, Theorem 2.1.16 states that  $Spin_n$  is a nontrivial twofold cover of  $SO_n$ . The reader should keep this fact in mind for future sections, since it helps to quickly characterize certain Spin groups. For example, the well-known diffeomorphism  $SO_3 \approx \mathbb{RP}^3$  implies that  $Spin_3 \approx S^3$ .

### 2.1.3 Spin representation

The application of Clifford algebras, and more particularly, of  $Spin_n$ , to this work does not come in a direct, clean-cut way. A basic understanding of their representations is necessary. With this in mind, consider a quadratic vector space  $(V, q)$  over a field  $\mathbb{K}$ .

**Definition 2.1.17.** Let  $\mathbb{F}$  be a field such that  $\mathbb{K} \subset \mathbb{F}$ . An  $\mathbb{F}$ -representation of  $\text{Cl}(V, q)$  is a  $\mathbb{K}$ -algebra homomorphism

$$\lambda : \text{Cl}(V, q) \longrightarrow \text{End}_{\mathbb{F}}(W),$$

where  $W$  is a finite-dimensional vector space over  $\mathbb{F}$ , and it is called a  $\text{Cl}(V, q)$ -module over  $\mathbb{F}$ . The *dimension* of the representation is the dimension of  $W$ ,  $\dim_{\mathbb{F}}(W)$ .

One can define the operation  $\lambda(x)(w)$  between an element  $x \in \text{Cl}(V, q)$  and  $w \in W$ . When there is no confusion, the notation can be simplified to  $x \cdot w$ . This product is referred to as a *Clifford multiplication*. For more information on representations, see [5].

As was the case in the previous section, this analysis will focus on  $\text{Cl}_n$  and its complex analogue,  $\text{Cl}_n^{\mathbb{C}} = \text{Cl}_n \otimes_{\mathbb{R}} \mathbb{C}$ .

**Definition 2.1.18.** Consider the following complex vector space:

$$\Delta_n := \mathbb{C}^{2^k} \cong \bigotimes_k \mathbb{C}^2, \quad n = 2k, 2k + 1.$$

This structure is known as the *vector space of complex  $n$ -spinors*. Its elements are called *complex spinors*.

It can be proved [12, p. 13] that

$$\text{Cl}_n^{\mathbb{C}} \cong \begin{cases} \text{End}_{\mathbb{C}}(\Delta_n) & \text{if } n \text{ is even} \\ \text{End}_{\mathbb{C}}(\Delta_n) \oplus \text{End}_{\mathbb{C}}(\Delta_n) & \text{if } n \text{ is odd.} \end{cases}$$

As a result, one can define the *spin representation*  $\delta_n$  of  $\text{Cl}_n^{\mathbb{C}}$ . In the even case, this is nothing but the isomorphism  $\delta_n : \text{Cl}_n^{\mathbb{C}} \xrightarrow{\cong} \text{End}_{\mathbb{C}}(\Delta_n)$ . The odd case is a little more convoluted, since one needs to compose the isomorphism with the projection onto the first component:

$$\delta_n : \text{Cl}_n^{\mathbb{C}} \xrightarrow{\cong} \text{End}_{\mathbb{C}}(\Delta_n) \oplus \text{End}_{\mathbb{C}}(\Delta_n) \xrightarrow{p_1} \text{End}_{\mathbb{C}}(\Delta_n).$$

Now, consider  $\text{Cl}_n$  as a subset of  $\text{Cl}_n^{\mathbb{C}}$ . By taking  $\text{Spin}_n \subset \text{Cl}_n \subset \text{Cl}_n^{\mathbb{C}}$ , one can restrict the representation  $\delta_n$  to obtain the spin representation of  $\text{Spin}_n$ ,

$$\delta_n : \text{Spin}_n \longrightarrow \text{Aut}(\Delta_n) \equiv \text{GL}(\Delta_n).$$

As usual, the same symbol is used to denote both the original mapping and its restriction.

**Proposition 2.1.19.** *The spin representation of  $\text{Spin}_n$  is faithful (i.e.  $\delta_n$  is injective).*

*Proof.* If  $n$  is even, then  $\text{Cl}_n^{\mathbb{C}} \cong \text{End}(\Delta_n)$ , so the statement is trivial. Moving on to

$n = 2k + 1$ , as vector spaces,  $\Delta_{2k} = \Delta_{2k+1}$ , and  $\text{Spin}_{2k}$  can be thought of as a subset of  $\text{Spin}_{2k+1}$ . Hence the diagram

$$\begin{array}{ccc} \text{Spin}_{2k} & \xrightarrow{\delta_{2k}} & \text{GL}(\Delta_{2k}) \\ \downarrow i & & \parallel \\ \text{Spin}_{2k+1} & \xrightarrow{\delta_{2k+1}} & \text{GL}(\Delta_{2k+1}) \end{array}$$

commutes. This implies that the normal subgroup  $H := \ker(\delta_{2k+1})$  satisfies

$$H \cap \text{Spin}_{2k} = \{1\}.$$

Now, elements  $\tilde{A} \in \text{SO}_{2k}$  can be identified with elements of the form  $\text{diag}(\tilde{A}, 1) \in \text{SO}_{2k+1}$ . Given that  $\rho$  is surjective,  $\rho(H) \subset \text{SO}_{2k+1}$  is normal. Moreover

$$\rho(H) \cap \text{SO}_{2k} = \{I\}.$$

Choose  $A \in \rho(H) \subset \text{SO}_{2k+1}$ . Its characteristic polynomial has odd degree, so either 1 or  $-1$  is a root. Since for an orientation-preserving matrix, both complex roots and  $-1$  can only appear in pairs, 1 has to be a root<sup>3</sup>, so there is a unit vector  $v$  such that  $A(v) = v$ . Thus there is an element  $B \in \text{SO}_{2k+1}$  such that  $BAB^{-1} \in \text{SO}_{2k}$ . Since  $\rho(H)$  is normal, this implies  $BAB^{-1} = I$ , hence  $\rho(H) = \{I\}$ . Since  $-1 \notin H$ , this leaves only the option  $H = \{1\}$ , so  $\delta_{2k+1}$  is injective.  $\blacksquare$

This proposition is not as unassuming as it might look. It actually implies that the spin representations do not arise as lifts of representations of  $\text{SO}_n$ , since such a lift necessarily contains  $\{\pm 1\}$  in its kernel.

Going back to the concept of Clifford multiplications (Definition 2.1.17), fix a vector  $x \in \mathbb{R}^n \subset \text{Cl}_n \subset \text{Cl}_n^{\mathbb{C}}$ . Via the representation  $\delta_n$ ,  $x$  can be considered as an endomorphism of  $\Delta_n$ . Since  $x$  is arbitrary, the *Clifford multiplication of vectors and spinors* can be defined as the linear map

$$\mu : \mathbb{R}^n \otimes_{\mathbb{R}} \Delta_n \longrightarrow \Delta_n,$$

where  $\mu(x \otimes \psi) = \delta_n(x)(\psi) \equiv x \cdot \psi$  for  $x \in \mathbb{R}^n$ ,  $\psi \in \Delta_n$ . Furthermore, this map can be extended to the exterior algebra  $\bigwedge^{\bullet} \mathbb{R}^n$ . Let  $(e_1, \dots, e_n)$  be a positively oriented orthonormal basis of  $\mathbb{R}^n$ . For a multivector  $w \in \bigwedge^{\bullet} \mathbb{R}^n$ ,

$$w = \sum_{i_1 < \dots < i_k} w_{i_1 \dots i_k} e_{i_1} \wedge \dots \wedge e_{i_k},$$

---

<sup>3</sup>This is clear by looking at the canonical form of orthogonal transformations. Geometrically, by the Cartan-Dieudonné Theorem,  $A \in \text{SO}_{2k+1}$  corresponds to the composition of, at most,  $2k$  reflections across hyperplanes, so there is always a vector in the intersection of such hyperplanes that remains invariant by the reflections.

define

$$\mu(w \otimes \psi) = w \cdot \psi = \sum_{i_1 < \dots < i_k} w_{i_1 \dots i_k} e_{i_1} \cdots e_{i_k} \cdot \psi.$$

It can be shown [12, pp. 21-22] that  $\mu$  is a homomorphism of representations of  $\text{Spin}_n$ , and that for  $x \in \mathbb{R}^n$  and  $w \in \bigwedge^\bullet \mathbb{R}^n$ , one has the equality

$$x \cdot (w \cdot \psi) = (x^\flat \wedge w) \cdot \psi - (x \lrcorner w) \cdot \psi,$$

where  $\lrcorner$  denotes the interior product. Since  $\psi$  is arbitrary, the spinor is usually dropped and, making use of the isomorphism  $\text{Cl}_n \cong \bigwedge^\bullet \mathbb{R}^n$ , the following equation is used to characterize the Clifford product by vectors  $x \in \mathbb{R}^n \subset \text{Cl}_n$ ,

$$x \cdot \equiv x^\flat \wedge - x \lrcorner. \quad (2.1.5)$$

The reader should keep (2.1.5) in mind, since it will be used in future sections.

This knowledge about the Clifford multiplication can be used to delve deeper into the properties of the spin representation. Take the so-called “complex volume form” of  $\mathbb{R}^n$ ,

$$\omega_n^{\mathbb{C}} = i^{\lfloor (n+1)/2 \rfloor} e_1 \cdots e_n.$$

In the case  $n = 2k$ , the endomorphism

$$f := \delta_{2k}(\omega_{2k}^{\mathbb{C}}) : \Delta_{2k} \longrightarrow \Delta_{2k}$$

satisfies  $f(\delta_{2k}(g)(\psi)) = \delta_{2k}(g)(f(\psi))$ , since  $x \cdot \omega_{2k}^{\mathbb{C}} = \omega_{2k}^{\mathbb{C}} \cdot \alpha(x)$ , for every  $x \in \text{Cl}_n$  [8, p. 22] (so  $\omega_{2k}^{\mathbb{C}}$  is central in  $\text{Cl}_n^0$ ). Hence it is an automorphism of the representation  $\delta_n$ . Moreover

$$(\omega_{2k}^{\mathbb{C}})^2 = i^{2k} (-1)^{k(2k-1)} e_1 \cdots e_{2k} \cdot e_{2k} \cdots e_1 = (-1)^k (-1)^{-k} (-1)^{2k} = 1,$$

so  $f^2 = \text{id}_{\Delta_n}$ . The spin representation decomposes into two subspaces,

$$\Delta_{2k} = \Delta_{2k}^+ \oplus \Delta_{2k}^-, \quad \Delta_{2k}^\pm := \{\psi \in \Delta_{2k} : f(\psi) = \pm \psi\}.$$

The spinors belonging to  $\Delta_{2k}^\pm$  are often referred to as *positive and negative Weyl spinors*, respectively. The representation  $\delta_n$  splits into  $\delta_n = \delta_n^+ \oplus \delta_n^-$ .

The next step of this analysis is proving some of the properties of  $\Delta_{2k+1}$  and  $\Delta_{2k}^\pm$ . This will culminate in the proof that the representations  $\delta_{2k+1}$  and  $\delta_{2k}^\pm$  are irreducible.

**Proposition 2.1.20.** *The following affirmations regarding the spaces  $\Delta_n$  hold:*

- (i)  $\dim_{\mathbb{C}}(\Delta_{2k+1}) = 2^k$ .
- (ii)  $\dim_{\mathbb{C}}(\Delta_{2k}^\pm) = 2^{k-1}$ .
- (iii) Let  $x \in \mathbb{R}^{2k}$  and  $\psi^\pm \in \Delta_{2k}^\pm$ . Then  $x \cdot \psi^\pm \in \Delta_{2k}^\mp$ , so Clifford multiplication induces homomorphisms  $\mu : \mathbb{R}^{2k} \otimes_{\mathbb{R}} \Delta_{2k}^\pm \rightarrow \Delta_{2k}^\mp$ .

*Proof.* Property (i) is clear by the definition of  $\Delta_{2k+1}$ . Property (iii) follows from the following Clifford relation:

$$x \cdot e_1 \cdots e_{2k} = e_1 \cdots e_{2k} \cdot \alpha(x) = -e_1 \cdots e_{2k} \cdot x,$$

for  $x \in \mathbb{R}^n \setminus \{0\}$ . This implies that Clifford multiplication by any vector anti-commutes with  $f$ , thus it maps  $\Delta_{2k}^\pm$  onto  $\Delta_{2k}^\mp$  in a bijective manner. Consequently, both subspaces have the same dimension, and  $\dim(\Delta_{2k}^\pm) = 2^k/2 = 2^{k-1}$ , obtaining property (ii). ■

The irreducibility of these representations is a little trickier, since it requires some algebraic notions that, albeit well-known, have not been explored here. For the sake of simplicity, the following lemma, which is key to subsequent propositions, will be cited without proof.

**Lemma 2.1.21** (As seen in [12], p. 23). *Let  $V, W$  be complex vector spaces such that  $\dim(W) < \dim(V)$ . Then there are no nontrivial algebra homomorphisms  $\text{End}(V) \rightarrow \text{End}(W)$ .*

**Proposition 2.1.22.** *The representations  $\delta_{2k}^\pm$  are irreducible, i.e., there are no proper subspaces which are invariant by the representation.*

*Proof.* The treatment is analogous for both subspaces, so only the positive case will be explained. Proceed by contradiction. Assume that  $\{0\} \neq W \subsetneq \Delta_{2k}^+$  is a proper invariant subspace. The Clifford products  $e_i e_j$  ( $i < j$ ) belong to  $\text{Spin}_{2k}$ , so  $W$  is invariant under Clifford multiplication by them ( $e_i e_j \cdot W \subset W$ ). Additionally, the set  $\{e_i e_j : i < j\}$  generates the even part of the complex Clifford algebra,  $(\text{Cl}_{2k}^\mathbb{C})^0$ . Hence there is a complex (nontrivial) representation

$$f : (\text{Cl}_{2k}^\mathbb{C})^0 \rightarrow \text{End}(W).$$

Given that  $(\text{Cl}_{2k}^\mathbb{C})^0 \cong \text{Cl}_{2k-1}^\mathbb{C} \cong \text{End}(\Delta_{2k-1}) \oplus \text{End}(\Delta_{2k-1})$  and that  $\dim(W) < \dim(\Delta_{2k-1})$ , by Lemma 2.1.21,  $f$  is trivial. This contradiction comes from assuming that  $W$  is a proper invariant subspace. ■

**Proposition 2.1.23.** *The representation  $\delta_{2k+1} : \text{Spin}_{2k+1} \rightarrow \Delta_{2k+1}$  is irreducible.*

*Proof.* The argument is analogous to the even case. Now,

$$\text{Spin}_{2k+1} \subset (\text{Cl}_{2k+1}^\mathbb{C})^0 \subset \text{Cl}_{2k+1}^\mathbb{C} \cong \text{End}(\Delta_{2k+1}) \oplus \text{End}(\Delta_{2k+1}).$$

Let  $\{0\} \neq W \subsetneq \Delta_{2k+1}$  be a proper invariant subspace. As before, it follows that  $W$  is also invariant by the action of  $(\text{Cl}_{2k+1}^\mathbb{C})^0$  (since it is generated by elements of  $\text{Spin}_{2k+1}$ ), so there is a nontrivial representation

$$f : (\text{Cl}_{2k+1}^\mathbb{C})^0 \longrightarrow \text{End}(W).$$

Now  $(\text{Cl}_{2k+1}^{\mathbb{C}})^0 \cong \text{Cl}_{2k}^{\mathbb{C}} \cong \text{End}(\Delta_{2k})$ . Since  $\dim(W) < \dim(\Delta_{2k+1}) = \dim(\Delta_{2k})$ , by Lemma 2.1.21,  $f$  is once again trivial.  $\blacksquare$

## 2.2 The spinor bundle

In this section, all the previously discussed geometrical and algebraic notions come together to finally define the structure on which Dirac operators act.

From now on, given an oriented Riemannian manifold  $(M^n, g)$ , let  $F_{\text{SO}}(TM)$  denote the  $\text{SO}_n$ -principal bundle of its positively oriented orthonormal frames.

**Definition 2.2.1.** Let  $(M^n, g)$  be an oriented Riemannian manifold of dimension  $n$ . A *spin structure* on  $(M^n, g)$  is a  $\text{Spin}_n$ -principal bundle  $\text{Spin}(TM) \rightarrow M$ , together with a twofold covering map  $\text{Spin}(TM) \xrightarrow{\eta} F_{\text{SO}}(TM)$  compatible with the respective group actions, i.e., such that the diagram

$$\begin{array}{ccc} \text{Spin}(TM) \times \text{Spin}_n & \longrightarrow & \text{Spin}(TM) \\ \downarrow \eta \times \rho & & \downarrow \eta \\ F_{\text{SO}}(TM) \times \text{SO}_n & \longrightarrow & F_{\text{SO}}(TM) \end{array} \quad \begin{array}{c} \nearrow \\ M \\ \nwarrow \end{array}$$

commutes. A *spin manifold* is an oriented Riemannian manifold with a spin structure.

Not every manifold is spin. The conditions for the existence of a spin structure on a manifold  $(M^n, g)$  are of topological nature [13, p. 2]. Henceforth, when referring to a manifold  $M$ , it will be assumed that it is spin.

**Definition 2.2.2.** The *spinor bundle* of a manifold  $M$  is the complex vector bundle associated to  $\text{Spin}(TM)$  via the spin representation,

$$\Sigma M = \text{Spin}(TM) \times_{\delta_n} \Delta_n.$$

Recall that, for  $n$  even,  $\Delta_n = \Delta_n^+ \oplus \Delta_n^-$  and  $\delta_n = \delta_n^+ \oplus \delta_n^-$ . Hence the *positive and negative spinor bundles* can be defined as

$$\Sigma M^{\pm} = \text{Spin}(TM) \times_{\delta_n^{\pm}} \Delta_n^{\pm},$$

so  $\Sigma M = \Sigma M^+ \oplus \Sigma M^-$ . The spinor bundle is thus a vector bundle with typical fiber  $\Delta_n$  (or  $\Delta_n^{\pm}$ , in the positive and negative cases). Elements of  $\Sigma M$  are known as *spinors*. Just

as sections of the tangent bundle,  $X \in \mathfrak{X}(M)$ , are called vector fields, sections of the spinor bundle,  $\psi \in \Gamma(\Sigma M)$ , are called *spinor fields*<sup>4</sup> or, sometimes, just spinors.

The Clifford multiplication of multivectors and spinors can be extended to a Clifford multiplication on the spinor bundle. One starts with the fiberwise action given by

$$T^*M \otimes \Sigma M \longrightarrow \Sigma M, \quad X^\flat \otimes \psi \mapsto X \cdot \psi := \delta_n(X)(\psi),$$

such that

$$X \cdot Y \cdot \psi + Y \cdot X \cdot \psi = -2g(X, Y)\psi. \quad (2.2.1)$$

Just like in Section 2.1.3, for a  $k$ -form  $w = \sum_{i_1 < \dots < i_k} w_{i_1 \dots i_k} e_{i_1}^* \wedge \dots \wedge e_{i_k}^*$ , (where  $(e_1, \dots, e_n)$  is a positively oriented orthonormal basis of  $TM$ ), the Clifford multiplication by  $\psi$  is

$$\mu(w \otimes \psi) = w \cdot \psi = \sum_{i_1 < \dots < i_k} w_{i_1 \dots i_k} e_{i_1} \cdots e_{i_k} \cdot \psi,$$

so the homomorphism extends to  $\bigwedge^\bullet TM \otimes \Sigma M$ .

*Remark 2.2.3.* Using the Riemannian metric on  $M$ , the orthonormal basis  $(e_1, \dots, e_n)$  and its dual basis,  $(e_1^*, \dots, e_n^*)$ , can be identified. For the sake of simplicity, when there is no confusion, the superscript  $*$  will be dropped and  $e_i$  will refer to both each vector and its dual covector.

**Proposition 2.2.4.** *There is a Hermitian structure  $\langle \cdot, \cdot \rangle$  on  $\Sigma M$  such that*

$$\langle X \cdot \varphi, \psi \rangle = -\langle \varphi, X \cdot \psi \rangle$$

for every  $X \in \mathfrak{X}(M)$  and  $\varphi, \psi \in \Gamma(\Sigma M)$ .

*Proof.* This holds because the spin representation is unitary; see e.g. [14, pp. 129-130]. ■

The spinor bundle is a versatile and intricate structure, and a much more in-depth analysis of it is possible (see, for example, [8, Chapter II]). This very work will eventually dive deeper into its properties. For now, however, it will be put on the back burner until Section 2.3.

## 2.3 Connections on spinor bundles

The description of a connection on the spinor bundle uses the concepts recalled in Section 1.2.2. If  $M$  is a spin manifold, the Levi-Civita connection on  $TM$  induces a connection

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<sup>4</sup>Spinor fields play an essential role in many fields of physics, such as particle physics or quantum mechanics.

on the  $\mathrm{SO}_n$ -principal bundle  $F_{\mathrm{SO}}(TM)$  of positively oriented frames. This connection can in turn be lifted to one on  $\mathrm{Spin}(TM)$ , via the covering map  $\eta : \mathrm{Spin}(TM) \rightarrow F_{\mathrm{SO}}(TM)$ . Finally, one can canonically define an induced vector bundle connection on the associated spinor bundle. This connection is, at its core, induced by the Levi-Civita connection on  $TM$ , so it shall be called *Levi-Civita spinorial connection*. See e.g. [18, pp. 42-45] for a more detailed exposition of this process.

From this point on, when encountering spinorial operators, tensors  $\dots$ , the reader should assume that they are spinorial Levi-Civita quantities. Thus the symbol  $\nabla$  will be used both for the spinorial connection and the Levi-Civita connection on  $TM$ . The following results come from [14, p.140]:

**Proposition 2.3.1.** *Take an orthonormal basis of  $(\sigma_1, \dots, \sigma_{2\lfloor n/2 \rfloor})$  of  $\Delta_n$ , and a positively oriented local orthonormal basis  $(e_1, \dots, e_n)$  of  $TM$ . Let  $(\psi_\alpha)_{1 \leq \alpha \leq \lfloor n/2 \rfloor}$  be any corresponding local spinorial frame, i.e.  $\psi_\alpha = [\tilde{s}, \sigma_\alpha]$  with  $\eta(\tilde{s}) = (e_1, \dots, e_n)$ . Then the following affirmations hold:*

1. *Locally, the spinorial covariant derivative is given by*

$$\nabla \psi_\alpha = \frac{1}{4} \sum_{i,j=1}^n g(\nabla e_i, e_j) e_i \cdot e_j \cdot \psi_\alpha. \quad (2.3.1)$$

2. *The spinorial curvature tensor  $R^\nabla$  can be expressed explicitly in terms of the Riemann curvature tensor  $R$  on  $TM$ . This expression is*

$$R_{X,Y}^\nabla \psi = \frac{1}{4} \sum_{i,j=1}^n g(R_{X,Y} e_i, e_j) e_i \cdot e_j \cdot \psi. \quad (2.3.2)$$

3. *The spinorial covariant derivative is compatible with the Clifford multiplication “ $\cdot$ ” and the Hermitian structure  $\langle \cdot, \cdot \rangle$ , i.e., for every  $X, Y \in \mathfrak{X}(M)$  and  $\psi \in \Gamma(\Sigma M)$ ,*

$$\begin{aligned} X \langle \psi, \varphi \rangle &= \langle \nabla_X \psi, \varphi \rangle + \langle \psi, \nabla_X \varphi \rangle, \\ \nabla_X (Y \cdot \psi) &= (\nabla_X Y) \cdot \psi + Y \cdot \nabla_X \psi. \end{aligned}$$



## Chapter 3

# The Dirac operator

By this point, an impatient reader might have begun to wonder when he is finally going to come across the titular Dirac operators. The answer lies in this chapter. In fact, he might be surprised by the apparent simplicity of their definition, once all the preliminary pieces and tools have been put in place. This simplicity, however, is deceiving. They are very general operators, that can be defined in highly involved vector bundles, so their properties (such as their spectrum) can be extremely challenging to analyze. This chapter will try to do so for some of the more essential ones.

### 3.1 Definition of the Dirac operator

For the sake of clarity, recap the three most important features associated to a spin structure on a Riemannian manifold  $(M^n, g)$ :

- (i) The spinor bundle  $\Sigma M = \text{Spin}(TM) \times_{\delta_n} \Delta_n$  with the fiberwise Clifford multiplication between  $k$ -forms and spinors (see Section 2.2).
- (ii) The natural Hermitian structure on  $\Sigma M$  (see Section 2.2).
- (iii) The Levi-Civita spinorial connection on  $\Sigma M$  (see Section 2.3).

These three structures satisfy three simple compatibility relations, for  $X, Y \in \mathfrak{X}(M)$  and  $\psi, \varphi \in \Gamma(\Sigma M)$ :

$$\langle X \cdot \psi, \varphi \rangle + \langle \psi, X \cdot \varphi \rangle = 0, \quad (3.1.1)$$

$$X \langle \psi, \varphi \rangle = \langle \nabla_X \psi, \varphi \rangle + \langle \psi, \nabla_X \varphi \rangle, \quad (3.1.2)$$

$$\nabla_X(Y \cdot \psi) = (\nabla_X Y) \cdot \psi + Y \cdot \nabla_X \psi. \quad (3.1.3)$$

By combining the spinorial connection and the Clifford multiplication, the central object of this review is finally within reach.

**Definition 3.1.1** (Fundamental Dirac operator). The *fundamental Dirac operator* (often just called Dirac operator) of a Riemannian spin manifold  $(M^n, g)$  is a map

$$D : \Gamma(\Sigma M) \longrightarrow \Gamma(\Sigma M)$$

defined, for every  $\psi \in \Gamma(\Sigma M)$ , by

$$D\psi = \sum_{i=1}^n e_i \cdot \nabla_{e_i} \psi.$$

where  $(e_i)_{1 \leq i \leq n}$  is any local orthonormal basis of  $TM$ .

*Remark 3.1.2.* Using the alternative definition of the spinorial connection given by Remark 1.2.1, the Dirac operator  $D$  can be regarded as the composition of the spinorial connection and Clifford multiplication, via the identity  $T^*M \equiv TM$  given by the metric,

$$\Gamma(\Sigma M) \xrightarrow{\nabla} \Gamma(T^*M \otimes \Sigma M) \equiv \Gamma(TM \otimes \Sigma M) \xrightarrow{\cdot} \Gamma(\Sigma M).$$

It follows that  $D$  is a first order differential operator (Example 1.4.2).

## 3.2 Basic properties of the Dirac operator

This section will start by proving a series of lemmas and propositions regarding general properties of the Dirac operator. They will later feed into more involved lines of reasoning regarding, for example, its spectrum.

Before diving into these lemmas, the reader should be aware of the following equations. Under the fiberwise isomorphism  $\text{Cl}(T_x M) \cong \bigwedge^\bullet(T_x M)$  that identifies

$$v \cdot \psi \equiv v \wedge \psi - v \lrcorner \psi, \quad (3.2.1)$$

the exterior differential,  $d$ , and its formal adjoint,  $\delta$  (the so-called codifferential), can be

locally written as [14, p. 148]

$$d = \sum_{i=1}^n e_i \wedge \nabla_{e_i}, \quad \delta = - \sum_{i=1}^n e_i \lrcorner \nabla_{e_i}, \quad (3.2.2)$$

so the Dirac operator becomes

$$D = \sum_{i=1}^n e_i \cdot \nabla_{e_i} \equiv d + \delta,$$

which establishes  $D$  as the “square root” of the *Laplace-de Rham operator*<sup>1</sup>  $\Delta_{LdR} = d\delta + \delta d$ ,

$$D^2 \equiv (d + \delta)^2 = d^2 + \delta^2 + d\delta + \delta d = d\delta + \delta d = \Delta_{LdR}.$$

With this in mind, it is time to start a proper study of  $D$ .

**Lemma 3.2.1.** *Let  $\psi$  be a smooth spinor field,  $f$  a smooth function and  $\xi$  a smooth vector field on a Riemannian spin manifold  $(M^n, g)$ . Let  $\text{grad}(f)$  be the gradient vector field of  $f$ , and  $d$  and  $\delta$  be the differential and codifferential, respectively. Then*

- (i)  $[D, f]\psi = D(f\psi) - fD\psi = df \cdot \psi = \text{grad}(f) \cdot \psi$ .
- (ii)  $D(\xi \cdot \psi) = -\xi \cdot D\psi - 2\nabla_\xi \psi + (d + \delta)\xi^\flat \cdot \psi$ .
- (iii)  $D^2(f\psi) = fD^2\psi - 2\nabla_{\text{grad}(f)}\psi + (\Delta f)\psi$ , where  $\Delta := \delta d$  is the scalar Laplace operator on  $(M^n, g)$ .

*Proof.* Choose a local orthonormal basis  $(e_j)_{1 \leq j \leq n}$  of  $TM$ . Then

$$\begin{aligned} D(f\psi) - fD\psi &= \sum_{i=1}^n e_i \cdot \nabla_{e_i}(f\psi) - fD\psi \\ &= \sum_{i=1}^n df(e_i)e_i \cdot \psi + fD\psi - fD\psi = df \cdot \psi. \end{aligned}$$

This, together with the identification  $TM \equiv T^*M$  (hence  $\text{grad}(f) \cdot \equiv df \cdot$ ), proves prop-

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<sup>1</sup>This was actually what Paul Dirac was looking for when studying relativistic wave functions. He wanted to find a relativistic analogue to the Schrödinger equation that was linear in the time derivatives, which would enable him to extend the probabilistic interpretation of non-relativistic wave functions to the relativistic case. As it turns out, the Dirac operator does the job.

erty (i). Moving on to property (ii),

$$\begin{aligned}
D(\xi \cdot \psi) &= \sum_{i=1}^n e_i \cdot \nabla_{e_i}(\xi \cdot \psi) = \sum_{i=1}^n e_i \cdot (\nabla_{e_i} \xi) \cdot \psi + \sum_{i=1}^n e_i \cdot \xi \cdot \nabla_{e_i} \psi \\
&\stackrel{(2.2.1)}{=} \sum_{i=1}^n e_i \cdot (\nabla_{e_i} \xi) \cdot \psi - \xi \cdot \sum_{i=1}^n e_i \cdot \nabla_{e_i} \psi - 2 \sum_{i=1}^n g(\xi, e_i) \nabla_{e_i} \psi \\
&\stackrel{(3.2.1)}{=} \left( \sum_{i=1}^n e_i \wedge \nabla_{e_i} \xi^b \right) \cdot \psi - \left( \sum_{i=1}^n e_i \lrcorner \nabla_{e_i} \xi^b \right) \cdot \psi \\
&\quad - \xi \cdot \sum_{i=1}^n e_i \cdot \nabla_{e_i} \psi - 2 \sum_{i=1}^n g(\xi, e_i) \nabla_{e_i} \psi \\
&\stackrel{(3.2.2)}{=} -\xi \cdot D\psi - 2\nabla_\xi \psi + (d + \delta)\xi^b \cdot \psi.
\end{aligned}$$

This proves property (ii). For property (iii),

$$\begin{aligned}
D^2(f\psi) &\stackrel{(i)}{=} D(df \cdot \psi + fD\psi) \\
&\stackrel{(i) \& (ii)}{=} -df \cdot D\psi - 2\nabla_{\text{grad}(f)} \psi + (d + \delta)df \cdot \psi + df \cdot D\psi + fD^2\psi \\
&= fD^2\psi - 2\nabla_{\text{grad}(f)} \psi + (\Delta f)\psi.
\end{aligned}$$

■

**Proposition 3.2.2.** *The Dirac operator is an elliptic operator.*

*Proof.* Let  $x \in M$ ,  $\xi \in T_x^*M \setminus \{0\}$  and  $f \in C^\infty(M)$  such that  $(df)_x = \xi$ . Consider the following expression:

$$D[(f - f(x))\psi](x),$$

where  $\psi$  is a smooth spinor field. It is clear that only the highest order terms survive the subtraction. Thus it can be identified with the principal symbol in the following manner:

$$\begin{aligned}
\sigma_\xi(D)(\psi(x)) &= D[(f - f(x))\psi](x) \\
&= (fD\psi + df \cdot \psi - f(x)D\psi)(x) = (df)_x \cdot \psi(x) = \xi \cdot \psi(x),
\end{aligned}$$

where Lemma 3.2.1 (i) has been used. In other words  $\sigma_\xi(D) \in \text{End}(\Sigma_x M)$  is  $\xi \cdot$  (Clifford multiplication by  $\xi$ ). Then the ellipticity of  $D$  means that  $\xi \cdot$  is an isomorphism of  $\Sigma_x M$  for every possible  $\xi \in T_x^*M \setminus \{0\}$ . Indeed, using the identity  $T_x^*M \equiv T_x M$ ,

$$\xi \cdot \psi = 0 \implies \xi \cdot \xi \cdot \psi = 0 \stackrel{(2.2.1)}{\iff} -|\xi|^2 \psi = 0 \iff \psi = 0. \quad \blacksquare$$

**Proposition 3.2.3.** *If  $M$  is a closed<sup>2</sup> manifold, then the Dirac operator is formally self-adjoint with respect to  $\langle \cdot, \cdot \rangle_{L^2}$ .*

*Proof.* Recall the compatibility conditions (3.1.1)-(3.1.3) and choose, at each point  $x \in M$ , a synchronous local orthonormal frame  $(e_i)_{1 \leq i \leq n}$ , that is, a local frame around  $x$  such that

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<sup>2</sup>A compact manifold without boundary.

$(\nabla_{e_i} e_j)(x) = 0$ , for  $1 \leq i, j \leq n$  (this can always be done due to the symmetric nature of the Levi-Civita connection [17, pp. 146-148]). Then

$$\begin{aligned} \langle D\psi, \varphi \rangle(x) &= \left\langle \sum_{i=1}^n e_i \cdot \nabla_{e_i} \psi, \varphi \right\rangle(x) = - \sum_{i=1}^n \langle \nabla_{e_i} \psi, e_i \cdot \varphi \rangle(x) \\ &= - \sum_{i=1}^n [e_i \langle \psi, e_i \cdot \varphi \rangle - \langle \psi, \nabla_{e_i} (e_i \cdot \varphi) \rangle](x) \\ &= - \sum_{i=1}^n e_i \langle \psi, e_i \cdot \varphi \rangle(x) + \langle \psi, D\varphi \rangle(x). \end{aligned}$$

The intuitive idea behind this proof is showing that the sum in the last term is the divergence of a complex vector field, so its integral over  $M$  vanishes. This is thanks to the Stokes theorem for compactly supported vector fields, since  $\partial M = \emptyset$  because  $M$  is closed. To that end, consider two vector fields  $X_1, X_2 \in \mathfrak{X}(M)$ , defined for every  $Y \in T_x M$  by

$$[g(X_1, Y) + i g(X_2, Y)](x) = \langle \psi, Y \cdot \varphi \rangle(x).$$

By the properties of the Levi-Civita connection and the normal coordinates,

$$\begin{aligned} [\operatorname{div}(X_1) + i \operatorname{div}(X_2)](x) &= \sum_{k=1}^n [g(\nabla_{e_k} X_1, e_k) + i g(\nabla_{e_k} X_2, e_k)](x) \\ &= \sum_{k=1}^n [e_k(g(X_1, e_k) + i g(X_2, e_k))](x) = \sum_{k=1}^n e_k \langle \psi, e_k \cdot \varphi \rangle(x). \end{aligned}$$

So the following coordinate independent equality holds:

$$\langle D\psi, \varphi \rangle = -\operatorname{div}(X_1) - i \operatorname{div}(X_2) + \langle \psi, D\varphi \rangle,$$

and it can be integrated over  $M$  to prove the formal self-adjointness of  $D$ :

$$\int_M \langle D\psi, \varphi \rangle \nu_g = \int_M \langle \psi, D\varphi \rangle \nu_g. \quad \blacksquare$$

*Remark 3.2.4.* The reader should be aware that Proposition 3.2.3 does not prove the self-adjointness of  $D$  as a differential operator on  $\Sigma M$ , it only proves that it is symmetric. Self-adjointness is far more finicky and subtle, since  $D$  and its adjoint  $D^*$  are not necessarily defined over the same domain (among other technical fiddles). For complete manifolds, and thus for closed manifolds, symmetric differential operators are essential self-adjoint [13, pp. 12-13], but this work shall not go into it.

**Proposition 3.2.5.** *Let  $(M^n, g)$  be an even dimensional spin manifold. Then its Dirac operator splits into*

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix},$$

using the decomposition  $\Gamma(\Sigma M) = \Gamma(\Sigma M^+) \oplus \Gamma(\Sigma M^-)$ , where  $D^\pm : \Gamma(\Sigma M^\pm) \rightarrow \Gamma(\Sigma M^\mp)$ , i.e. the Dirac operator maps positive spinors to negative ones and vice versa.

*Proof.* Recall that, for  $n = 2k$ , the complex spinor space decomposes into

$$\Delta_{2k} = \Delta_{2k}^+ \oplus \Delta_{2k}^-, \quad \Delta_{2k}^\pm := \{\psi \in \Delta_{2k} : \omega_{2k}^\mathbb{C} \cdot \psi = \pm \psi\}.$$

Since each fiber of  $\Sigma M$  is isomorphic to  $\Delta_{2k}$ , the action of the complex volume form translates to an action on  $\Gamma(\Sigma M)$ . Now, take a positive spinor field  $\psi \in \Gamma(\Sigma M^+)$ . Then

$$\begin{aligned} \omega_{2k}^\mathbb{C} \cdot D\psi &= \omega_{2k}^\mathbb{C} \cdot \sum_{i=1}^n e_i \cdot \nabla_{e_i} \psi = - \sum_{i=1}^n e_i \cdot \omega_{2k}^\mathbb{C} \cdot \nabla_{e_i} \psi \\ &= - \sum_{i=1}^n e_i \cdot \nabla_{e_i} (\omega_{2k}^\mathbb{C} \cdot \psi) = -D\psi. \end{aligned}$$

The treatment is analogous for negative spinors. ■

**Corollary 3.2.6.** *If  $(M^n, g)$  is an even dimensional closed spin manifold, the eigenvalues of  $D$  are symmetric with respect to zero.*

*Proof.* Suppose  $\psi$  is an eigenspinor for  $D$ , i.e.  $D\psi = \lambda\psi$  for  $\lambda \in \mathbb{R}$  ( $\lambda$  is guaranteed to be real, and  $\psi$  to be a smooth spinor field, by Theorem 1.4.4). Use the decomposition  $\Sigma M = \Sigma M^+ \oplus \Sigma M^-$  to write  $\psi = \psi^+ + \psi^-$ . By Proposition 3.2.5, it is clear that  $D\psi^\pm = \lambda\psi^\mp$ . Define a new spinor field  $\tilde{\psi} := \psi^+ - \psi^-$ . Then,

$$D\tilde{\psi} = D(\psi^+ - \psi^-) = \lambda\psi^- - \lambda\psi^+ = -\lambda(\psi^+ - \psi^-) = -\lambda\tilde{\psi},$$

so  $\pm\lambda \in \text{Spec}(D)$ . ■

*Remark 3.2.7.* Using the existence of real or quaternionic  $\text{Spin}_n$ -equivariant structures  $\alpha_n : \Delta_n \rightarrow \Delta_n$  anti-commuting with the Clifford multiplication [12, p. 31],

$$\alpha_n(x \cdot \psi) = -x \cdot \alpha_n(\psi), \quad x \in \mathbb{R}^n, \psi \in \Delta_n,$$

it can be shown that the symmetry of  $\text{Spec}(D)$  also holds for  $n \equiv 1 \pmod{4}$ . However, the symmetry can break for  $n \equiv 3 \pmod{4}$  [13, p. 16].

Before moving on to the next section, it is worthwhile to look at the simplest explicit example of the application of the Dirac operator to a spin manifold.

**Example 3.2.8** (Dirac operator on  $\mathbb{R}^n$ ). Let  $M = \mathbb{R}^n$ . Then,  $\Sigma\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{C}^{\lfloor n/2 \rfloor}$  is a trivial complex vector bundle. Therefore a spinor field  $\psi \in \Gamma(\Sigma\mathbb{R}^n)$  can be regarded as a smooth map  $\psi : \mathbb{R}^n \rightarrow \mathbb{C}^{\lfloor n/2 \rfloor}$ . The Dirac operator simplifies to

$$D = \sum_{i=1}^n e_i \cdot \partial_i$$

by identifying  $\partial_i = \nabla_{e_i}$ . Its square becomes

$$\begin{aligned} D^2 &= \left( \sum_{i=1}^n e_i \cdot \partial_i \right) \left( \sum_{j=1}^n e_j \cdot \partial_j \right) = \sum_{i,j=1}^n e_i \cdot e_j \cdot \partial_i \partial_j \\ &= - \sum_i \partial_i^2 + \sum_{i < j} e_i \cdot e_j \cdot \partial_i \partial_j + \sum_{i > j} e_i \cdot e_j \cdot \partial_i \partial_j \\ &= - \sum_i \partial_i^2 + \sum_{i < j} e_i \cdot e_j \cdot (\partial_i \partial_j - \partial_j \partial_i) = - \sum_i \partial_i^2, \end{aligned}$$

so the square of the Dirac operator acts on smooth maps from  $\mathbb{R}^n$  to  $\mathbb{C}^{[n/2]}$  like the usual Laplacian of  $\mathbb{R}^n$  (with a negative sign).

### 3.3 The Schrödinger-Lichnerowicz formula

Historically, and by construction, the Dirac operator was envisioned as a means to obtain some sort of “square root” of the Laplacian or one of its multiple generalizations (like the rough Laplacian or the Laplace-de Rham operator). As it turns out, the Schrödinger-Lichnerowicz formula establishes this connection by relating  $D^2$  to one of them, showing that they only differ in a very simple curvature expression.

**Definition 3.3.1** (Extension of the Hermitian product). The natural Hermitian product  $\langle \cdot, \cdot \rangle$  on  $\Gamma(\Sigma M)$  can be extended to  $\Gamma(T^*M \otimes \Sigma M)$  by

$$(\alpha \otimes \psi, \beta \otimes \varphi) = g(\alpha, \beta) \langle \psi, \varphi \rangle,$$

where the metric extends to covectors by the identity  $TM \equiv T^*M$ . At a point  $x \in M$ , for  $\omega, \eta \in T_x^*M \otimes \Sigma_x M$  and an orthonormal basis  $(e_1, \dots, e_n)$  of  $T_x M$ ,

$$\langle \omega, \eta \rangle = \sum_{i=1}^n \langle \omega(e_i), \eta(e_i) \rangle.$$

**Definition 3.3.2** (Bochner Laplacian). Consider a vector bundle  $E$  over a compact, oriented, Riemannian manifold  $(M, g)$ , endowed with a metric connection

$$\nabla : \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E).$$

Take its  $L^2$ -adjoint (also called the formal adjoint),

$$\nabla^* : \Gamma(T^*M \otimes E) \longrightarrow \Gamma(E),$$

such that  $\langle \nabla^* \varphi, \psi \rangle_{L^2} = \langle \varphi, \nabla \psi \rangle_{L^2}$  for every  $\varphi \in \Gamma(T^*M \otimes E)$  and  $\psi \in \Gamma(E)$ .

The *Bochner Laplacian* is a second order differential operator defined by

$$\Delta_B = \nabla^* \nabla,$$

which differs to the *rough Laplacian*  $\Delta_R = \text{tr}(\nabla^2)$  by a sign.

The titular formula is not particularly challenging to compute, but it does need some previous groundwork in the form of the two following lemmas.

**Lemma 3.3.3.** *Taking a synchronous local orthonormal frame  $(\nabla_{e_i} e_j)(x) = 0$  at each  $x \in M$ , where  $(M^n, g)$  is a closed Riemannian spin manifold, yields*

$$\Delta_B \psi = - \sum_{i=1}^n \nabla_{e_i} \nabla_{e_i} \psi$$

for every smooth spinor field  $\psi \in \Gamma(\Sigma M)$ .

*Proof.* Choose  $\psi, \varphi \in \Gamma(\Sigma M)$ . By the definition of  $\nabla^*$ ,

$$\langle \Delta_B \psi, \varphi \rangle_{L^2} = \langle \nabla^* \nabla \psi, \varphi \rangle_{L^2} = \langle \nabla \psi, \nabla \varphi \rangle_{L^2} = \sum_{i=1}^n \langle \nabla_{e_i} \psi, \nabla_{e_i} \varphi \rangle_{L^2}.$$

Now, proceeding as in Proposition 3.2.3, define two vector fields  $X_1, X_2 \in \mathfrak{X}(M)$  by

$$g(X_1, X) + i g(X_2, X) = \langle \psi, X \cdot \varphi \rangle$$

for all  $X \in TM$ . Thus for every  $x \in M$ ,

$$\text{div}(X_1) + i \text{div}(X_2) = \sum_{k=1}^n e_k \langle \psi, e_k \cdot \varphi \rangle.$$

Hence

$$\begin{aligned} \sum_{k=1}^n \langle \nabla_{e_k} \psi, \nabla_{e_k} \varphi \rangle &= \sum_{k=1}^n [e_k \langle \nabla_{e_k} \psi, \varphi \rangle - \langle \nabla_{e_k} \nabla_{e_k} \psi, \varphi \rangle] \\ &= \text{div}(X_1) + i \text{div}(X_2) - \sum_{k=1}^n \langle \nabla_{e_k} \nabla_{e_k} \psi, \varphi \rangle. \end{aligned}$$

Integration of this sum over  $M$  makes the divergence terms vanish, and the required condition for  $\nabla^*$  to be the formal adjoint of  $\nabla$  follows. ■

**Lemma 3.3.4.** *The Ricci tensor, considered as a symmetric endomorphism of the tangent bundle  $\text{Ric} : TM \rightarrow TM$  (see [12, p. 64]) of the Riemannian spin manifold  $(M^n, g)$  with local orthonormal frame  $(e_i)_{1 \leq i \leq n}$  satisfies, for every  $X \in TM$  and  $\psi \in \Sigma M$ ,*

$$\sum_{i=1}^n e_i \cdot R_{X, e_i}^\nabla \psi = \frac{1}{2} \text{Ric}(X) \cdot \psi,$$

where  $\text{Ric}(X) = - \sum_{i=1}^n R_{X, e_i} e_i$ .



*Proof.* Use the definition of the spinorial curvature and the first Bianchi identity to find

$$\begin{aligned} \sum_{i=1}^n e_i \cdot R_{X,e_i}^\nabla \psi &\stackrel{(2.3.2)}{=} \frac{1}{4} \sum_{i,j,k=1}^n g(R_{X,e_i} e_j, e_k) e_i \cdot e_j \cdot e_k \cdot \psi \\ &\stackrel{(1.3.1)}{=} -\frac{1}{4} \sum_{i,j,k=1}^n g(R_{e_i,e_j} X, e_k) e_i \cdot e_j \cdot e_k \cdot \psi - \frac{1}{4} \sum_{i,j,k=1}^n g(R_{e_j,X} e_i, e_k) e_i \cdot e_j \cdot e_k \cdot \psi. \end{aligned}$$

Rearranging indexes on the second sum yields

$$\sum_{i=1}^n e_i \cdot R_{X,e_i}^\nabla \psi = -\frac{1}{4} \sum_{i,j,k=1}^n g(R_{X,e_i} e_j, e_k) (e_j \cdot e_k \cdot e_i - e_j \cdot e_i \cdot e_k) \cdot \psi.$$

Now, apply (2.2.1) to simplify the Clifford multiplication of the basis vectors

$$\begin{aligned} e_j \cdot e_k \cdot e_i - e_j \cdot e_i \cdot e_k &= -e_j \cdot e_i \cdot e_k - 2\delta_{ik} e_j + e_i \cdot e_j \cdot e_k + 2\delta_{ij} e_k \\ &= 2e_i \cdot e_j \cdot e_k + 4\delta_{ij} e_k - 2\delta_{ik} e_j, \end{aligned}$$

where  $\delta_{ab}$  is the Kronecker delta. Hence

$$\begin{aligned} 3 \sum_{i=1}^n e_i \cdot R_{X,e_i}^\nabla \psi &= - \sum_{i,k=1}^n g(R_{X,e_i} e_i, e_k) e_k \cdot \psi + \frac{1}{2} \sum_{i,j=1}^n g(R_{X,e_i} e_j, e_i) e_j \cdot \psi \\ &= \sum_{k=1}^n g(\text{Ric}(X), e_k) e_k \cdot \psi + \frac{1}{2} \sum_{j=1}^n g(\text{Ric}(X), e_j) e_j \cdot \psi = \frac{3}{2} \text{Ric}(X) \cdot \psi. \blacksquare \end{aligned}$$

With this knowledge, the main result of this section immediately follows.

**Theorem 3.3.5** (Schrödinger-Lichnerowicz formula). *The Dirac operator  $D$  of a closed Riemannian spin manifold  $(M^n, g)$  satisfies*

$$D^2 = \Delta_B + \frac{S}{4} \text{id}, \quad (3.3.1)$$

where  $S = -\sum_{i=1}^n e_i \cdot \text{Ric}(e_i)$  is the scalar curvature of  $(M^n, g)$ .

*Proof.* At each  $x \in M$ , choose a synchronous local orthonormal frame. Recall that this

means  $(\nabla_{e_i} e_j)(x) = 0$ . For any smooth spinor field  $\psi$ , using (3.1.3) and (2.2.1),

$$\begin{aligned}
D^2\psi &= \sum_{i,j=1}^n e_i \cdot \nabla_{e_i} (e_j \cdot \nabla_{e_j} \psi) = \sum_{i,j=1}^n (e_i \cdot \nabla_{e_i} e_j \cdot \nabla_{e_j} \psi + e_i \cdot e_j \cdot \nabla_{e_i} \nabla_{e_j} \psi) \\
&= \sum_{i,j=1}^n e_i \cdot e_j \cdot \nabla_{e_i} \nabla_{e_j} \psi \\
&= - \sum_{i=1}^n \nabla_{e_i} \nabla_{e_i} \psi + \sum_{i < j}^n e_j \cdot e_i \cdot (\nabla_{e_j} \nabla_{e_i} - \nabla_{e_i} \nabla_{e_j}) \psi \\
&= \Delta_B \psi - \frac{1}{2} \sum_{i,j=1}^n e_i \cdot e_j \cdot R_{e_i, e_j}^\nabla \psi.
\end{aligned}$$

The second sum can be simplified further by invoking Lemma 3.3.4,

$$\sum_{i,j=1}^n e_i \cdot e_j \cdot R_{e_i, e_j}^\nabla \psi = \frac{1}{2} \sum_{i=1}^n e_i \cdot \text{Ric}(e_i) \cdot \psi = -\frac{S}{2} \psi.$$

Since  $\psi \in \Gamma(\Sigma M)$  was arbitrary, this completes the proof. ■

*Remark 3.3.6.* Since, to prove formal adjointness, it is enough to prove it for spinor fields with compact support on  $M$ , and which vanish on  $\partial M$  [18, p. 113], [13, p. 12], the Schrödinger-Lichnerowicz formula can be extended to general compact manifolds.

The Schrödinger-Lichnerowicz formula comes into play in the proof of many more advanced results. In this work, it will be used for the proof of some lower nonzero eigenvalue bounds.

### 3.4 Conformal covariance

Let  $(M^n, g)$  be a Riemannian spin manifold. It is natural to wonder how the Dirac operator changes under a conformal change of the metric  $\bar{g} := e^{2u}g$ , where  $u \in C^\infty(M, \mathbb{R})$ . The fixed spin structure on  $(M^n, g)$  induces a spin structure on  $(M^n, \bar{g})$ . Explicitly, an isomorphism between the two  $\text{SO}_n$ -principal bundles is

$$\begin{aligned}
G_u : F_{\text{SO}_g}(TM) &\rightarrow F_{\text{SO}_{\bar{g}}}(TM) \\
\{X_1, \dots, X_n\} &\mapsto \{e^{-u}X_1, \dots, e^{-u}X_n\}.
\end{aligned}$$

Note that the usual notation  $e_i$  has been changed to  $X_i$  for the sake of simplicity. The new spin structure is then defined by the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Spin}_g(TM) & \xrightarrow{\tilde{G}_u} & \mathrm{Spin}_{\bar{g}}(TM) \\
\downarrow \eta_g & & \downarrow \eta_{\bar{g}} \\
\mathrm{Fso}_g(TM) & \xrightarrow{G_u} & \mathrm{Fso}_{\bar{g}}(TM)
\end{array}$$

where all the arrows are compatible with the group action. This induces an isomorphism of the associated spinor bundles, given for a spinor  $\psi = [s, \sigma] \in \Sigma M$  by

$$\begin{aligned}
\tilde{G}_u : \Sigma_g M &\rightarrow \Sigma_{\bar{g}} M \\
\psi &\mapsto \bar{\psi} = [\tilde{G}_u(s), \sigma].
\end{aligned}$$

It is easy to check that this map is an isometry with respect to the Hermitian product  $\langle \cdot, \cdot \rangle$  on the spinor bundle. Combining this with the isometry  $T_g M \rightarrow T_{\bar{g}} M$  determined by  $X \mapsto \bar{X} := e^{-u} X$  yields a relation between the Clifford multiplications “ $\cdot$ ” and “ $\bar{\cdot}$ ”,

$$\bar{X} \bar{\cdot} \bar{\psi} = \overline{X \cdot \psi}$$

for each  $X \in \mathfrak{X}(M)$  and  $\psi \in \Gamma(\Sigma M)$ . On the other hand, the induced spinorial covariant derivative follows from an elemental (but somewhat lengthy) calculation starting from the spinorial covariant derivative for  $g$ , given in local coordinates by (2.3.1):

$$\bar{\nabla}_X \bar{\psi} = \bar{\nabla}_X \bar{\psi} - \frac{1}{2} \overline{X \cdot \mathrm{grad}(u) \cdot \psi} - \frac{1}{2} \overline{X(u) \psi}. \quad (3.4.1)$$

A more straightforward derivation of this equation (using connection forms) can be seen in [18, p. 70]. Using (3.4.1), the conformal covariance of the Dirac operator can be proved. Without going into much detail, suffice to say that a differential operator  $P$  on the spinor bundle is *conformally covariant* if there exist  $a, b \in \mathbb{R}$  such that

$$\bar{P}(e^{au} \bar{\psi}) = e^{bu} \bar{P}\bar{\psi}$$

for every conformally transformed metric  $\bar{g}$  and every spinor field  $\psi \in \Gamma(\Sigma M)$ .

**Proposition 3.4.1.** *Consider a Riemannian spin manifold  $(M^n, g)$  and its conformally modified analogue  $(M^n, \bar{g})$ . Then, their respective Dirac operators, denoted as  $D \equiv D_g$  and  $\bar{D} \equiv D_{\bar{g}}$ , are related in the following manner:*

$$\bar{D} \left( e^{-(n-1)u/2} \bar{\psi} \right) = e^{-(n+1)u/2} \bar{D}\bar{\psi}.$$

*Proof.* Fix a local orthonormal basis  $(X_i)_{1 \leq i \leq n}$  of  $(M^n, g)$ . It is obvious that taking

$(\bar{X}_i)_{1 \leq i \leq n} = (e^{-u} X_i)_{1 \leq i \leq n}$  provides a local orthonormal basis of  $(M^n, \bar{g})$  as well. Hence

$$\begin{aligned} \bar{D}\bar{\psi} &= \sum_{i=1}^n \bar{X}_i \cdot \bar{\nabla}_{\bar{X}_i} \bar{\psi} = \sum_{i=1}^n e^{-2u} X_i \cdot \bar{\nabla}_{X_i} \bar{\psi} \\ &= e^{-2u} \sum_{i=1}^n X_i \cdot \left[ \bar{\nabla}_{X_i} \bar{\psi} - \frac{1}{2} \overline{X_i \cdot \text{grad}(u) \cdot \psi} - \frac{1}{2} X_i(u) \bar{\psi} \right] \\ &= e^{-u} \sum_{i=1}^n \left[ \overline{X_i \cdot \nabla_{X_i} \psi} - \frac{1}{2} \overline{X_i \cdot X_i \cdot \text{grad}(u) \cdot \psi} - \frac{1}{2} X_i(u) \overline{X_i \cdot \psi} \right]. \end{aligned}$$

Now, knowing that  $X_i^2 \cdot = -1 \cdot$ , and that  $\sum_{i=1}^n X_i(u) X_i = \text{grad}(u)$ , it is clear that

$$\bar{D}\bar{\psi} = e^{-u} \left( \overline{D\psi} + \frac{n-1}{2} \overline{\text{grad}(u) \cdot \psi} \right). \quad (3.4.2)$$

From this equation and Lemma 3.2.1 (i), it follows that

$$\begin{aligned} \bar{D} \left( e^{-(n-1)u/2} \bar{\psi} \right) &= e^{-u} \left[ \overline{D(e^{-(n-1)u/2} \psi)} + \frac{n-1}{2} e^{-(n-1)u/2} \overline{\text{grad}(u) \cdot \psi} \right] \\ &= -e^{-u} \frac{n-1}{2} e^{-(n-1)u/2} \overline{\text{grad}(u) \cdot D\psi} + e^{-u} e^{-(n-1)u/2} \overline{D\psi} \\ &\quad + e^{-u} \frac{n-1}{2} e^{-(n-1)u/2} \overline{\text{grad}(u) \cdot D\psi} = e^{-(n+1)u/2} \overline{D\psi}. \quad \blacksquare \end{aligned}$$

Conformal covariance is another important feature of the Dirac operator. As seen in Section 4.2.1, it helps to improve certain usual lower eigenvalue bounds via the so-called ‘‘Hijazi inequality’’. To that end, it is convenient to define a new, also conformally covariant, operator, which relates to  $D$  in a useful way.

**Definition 3.4.2** (Penrose operator). The *Penrose (or twistor) operator* of a Riemannian spin manifold  $(M^n, g)$  is a map

$$P : \Gamma(\Sigma M) \longrightarrow \Gamma(T^*M \otimes \Sigma M) \longrightarrow \Gamma(\ker(\cdot))$$

where the first arrow is given by the spinorial connection, and the second is given by the orthogonal projection on the kernel of the Clifford multiplication. In local orthonormal coordinates  $(e_i)_{1 \leq i \leq n}$ , it takes the form

$$P\psi = \sum_{i=1}^n e_i^* \otimes \left( \nabla_{e_i} \psi + \frac{1}{n} e_i \cdot D\psi \right) = \nabla \psi + \frac{1}{n} \sum_{i=1}^n e_i^* \otimes e_i \cdot D\psi.$$

It is trivial to verify that  $P\psi \in \Gamma(\ker(\cdot))$ . The Penrose operator can also be defined in the direction of a vector  $X \in \mathfrak{X}(M)$ , as the map  $P_X : \Gamma(\Sigma M) \rightarrow \Gamma(\Sigma M)$  given by

$$P_X \psi = \nabla_X \psi + \frac{1}{n} X \cdot D\psi.$$

*Remark 3.4.3.* The Penrose operator is particularly useful because it allows for an optimal

decomposition of the gradient of a spinor. It is obvious that, for every nonzero  $\psi \notin \ker(D)$ ,

$$\frac{1}{n} \sum_{i=1}^n e_i^* \otimes e_i \cdot D\psi \in \ker(\cdot)^\perp,$$

so by the Pythagorean Theorem:

$$|\nabla\psi|^2 = |P\psi|^2 + \frac{1}{n}|D\psi|^2.$$

This equation is clearly also true when  $D\psi = 0$ , so it holds for all  $\psi \in \Gamma(\Sigma M)$ . This decomposition will be used in a later section.

Regarding the conformal behavior of the Penrose operator, the following statement holds.

**Proposition 3.4.4.** *Consider a Riemannian spin manifold  $(M^n, g)$  and its conformally modified analogue  $(M^n, \bar{g})$ . Then, their respective Penrose operators, denoted as  $P \equiv P_g$  and  $\bar{P} \equiv P_{\bar{g}}$ , are related in the following manner:*

$$\bar{P}(e^{u/2} \bar{\psi}) = e^{u/2} \bar{P}\bar{\psi}.$$

*Proof.* By definition, for any  $X \in \mathfrak{X}(M)$ ,

$$\bar{P}_X(e^{u/2} \bar{\psi}) = \bar{\nabla}_X(e^{u/2} \bar{\psi}) + \frac{1}{n} X \lrcorner \bar{D}(e^{u/2} \bar{\psi}).$$

The analysis is more tractable when one separates both terms in this sum. Thus

$$\bar{\nabla}_X(e^{u/2} \bar{\psi}) = e^{u/2} \left[ \bar{\nabla}_X \bar{\psi} + \frac{1}{2} X(u) \bar{\psi} \right] \stackrel{(3.4.1)}{=} e^{u/2} \left[ \bar{\nabla}_X \bar{\psi} - \frac{1}{2} X \cdot \overline{\text{grad}(u) \cdot \psi} \right],$$

and, secondly,

$$\begin{aligned} \bar{D}(e^{u/2} \bar{\psi}) &= e^{u/2} \bar{D} \bar{\psi} + \frac{1}{2} e^{-u/2} \overline{\text{grad}(u) \cdot \psi} \\ &\stackrel{(3.4.2)}{=} e^{-u/2} \left[ \overline{D\psi} + \frac{n-1}{2} \overline{\text{grad}(u) \cdot \psi} \right] + \frac{1}{2} e^{-u/2} \overline{\text{grad}(u) \cdot \psi} \\ &= e^{-u/2} \left[ \overline{D\psi} + \frac{n}{2} \overline{\text{grad}(u) \cdot \psi} \right]. \end{aligned}$$

Combining both terms and using  $X \lrcorner \bar{\psi} = e^u \overline{X \cdot \psi}$  one gets

$$\begin{aligned} \bar{P}_X(e^{u/2} \bar{\psi}) &= e^{u/2} \left[ \bar{\nabla}_X \bar{\psi} - \frac{1}{2} X \cdot \overline{\text{grad}(u) \cdot \psi} \right] + e^{-u/2} X \lrcorner \left[ \frac{1}{n} \overline{D\psi} + \frac{1}{2} \overline{\text{grad}(u) \cdot \psi} \right] \\ &= e^{u/2} \left[ \bar{\nabla}_X \bar{\psi} + \frac{1}{n} X \cdot \overline{D\psi} \right] = e^{u/2} \bar{P}_X \bar{\psi}. \quad \blacksquare \end{aligned}$$

With all the basic properties of the Dirac operator in mind, the reader is now ready to dive into a small survey of the Dirac spectrum on compact manifolds in Chapter 4.



## Chapter 4

# The Dirac spectrum on compact manifolds

Study of the Dirac spectrum in general spin manifolds is very complex. Most of them, in fact, do not allow for an explicit calculation of it, while others only admit an analytical determination of some of their eigenvalues. This chapter is devoted to the study of the Dirac spectrum and it is divided in two fronts. First, explicit calculations for some simple cases are presented. The second part is more general, and it looks over some of the most well-known lower nonzero eigenvalue bounds in compact manifolds. Nonetheless, aside from a brief commentary on compact manifolds with boundary in Section 4.2.2, all of the spin manifolds in this chapter will be closed (compact without boundary).

Before tackling any particular examples, it is worthwhile to prove some general features of the Dirac spectrum, shared by all closed Riemannian spin manifolds.

**Theorem 4.0.1** (General properties of the Dirac spectrum). *Let  $(M^n, g)$  be a closed Riemannian spin manifold. The following statements about the spectrum  $\text{Spec}(D)$  of the Dirac operator of  $M$  hold:*

- (i)  $\text{Spec}(D)$  is a discrete, unbounded on both sides, subset of  $\mathbb{R}$ , and it is symmetric about the origin if  $n \not\equiv 3 \pmod{4}$ .
- (ii) Each eigenspace  $E_\lambda$  of  $D$  is finite-dimensional and consists of smooth spinor fields.
- (iii) The eigenspaces of  $D$  provide a complete orthonormal system for  $L^2(\Sigma M)$ ,

$$L^2(\Sigma M) = \overline{\bigoplus_{\lambda} E_{\lambda}}.$$

*Proof.* Since the Dirac operator is elliptic and essentially self-adjoint (Remark 3.2.4), Theorem 1.4.4 applies to the closure of  $D$ . Combining this with Corollary 3.2.6 and its subsequent remark proves all claims except for the unboundedness of  $\text{Spec}(D)$ . Proof of this last statement is not very informative, but a version of it can be seen in [13, pp. 15-16]. ■

Moving on, the following definition will be useful for some explicit calculations.

**Definition 4.0.2.** A *Killing spinor* on a Riemannian spin manifold  $(M^n, g)$  is a spinor field  $\psi \in \Gamma(\Sigma M)$  which satisfies

$$\nabla_X \psi = \lambda X \cdot \psi$$

for every  $X \in \mathfrak{X}(M)$ , where  $\lambda \in \mathbb{C}$  is a constant known as the *Killing number* of  $\psi$ . If  $\lambda = 0$ ,  $\psi$  is said to be a *parallel spinor*.

*Remark 4.0.3.* Note that every Killing spinor is an eigenspinor of the Dirac operator, but general eigenspinors need not be Killing spinors. In particular, every parallel spinor is also an eigenspinor of the eigenvalue 0, also known as a *harmonic spinor*.

## 4.1 Explicit computations of spectra

In this section, some of the few closed spin manifolds whose spectrum can be completely or partially determined will be presented.

### 4.1.1 Flat tori

Let  $\Gamma$  be a lattice, i.e. a discrete, cocompact, additive subgroup in  $\mathbb{R}^n$ . The *flat torus*  $T^n$  is defined to be the compact quotient  $T^n = \Gamma \backslash \mathbb{R}^n$  by the action of  $\Gamma$ . This manifold can be endowed with a flat metric, induced by the flat Euclidean metric of  $\mathbb{R}^n$ .

Henceforth, let  $(\gamma_1, \dots, \gamma_n)$  be a basis of  $\Gamma$ , and  $(\gamma_1^*, \dots, \gamma_n^*)$  its dual basis of the dual lattice,  $\Gamma^* := \{\theta \in (\mathbb{R}^n)^* : \theta(\Gamma) \subset \mathbb{Z}\}$ .

It is a known fact [19] that there are  $2^n$  different spin structures on  $T^n$ . They are given by the  $n$ -tuples  $(\delta_1, \dots, \delta_n)$  where  $\delta_i \in \{0, 1\}$  ( $i = 1, \dots, n$ ), which in turn come from the  $2^n$  group homomorphisms  $\epsilon_{\delta_1, \dots, \delta_n} : \Gamma \rightarrow \{\pm 1\}$ , defined by  $\epsilon_{\delta_1, \dots, \delta_n}(\gamma_i) = (-1)^{\delta_i}$ .

Furthermore, the spinor bundle  $\Sigma T^n$  preserves the Clifford multiplication and the covariant derivative from  $\Sigma \mathbb{R}^n$ , and the smooth sections of  $\Sigma T^n$  can be identified with the



$\Gamma$ -equivariant smooth sections of  $\Sigma\mathbb{R}^n$  [13, p. 21], i.e.

$$\Gamma(\Sigma T^n) \equiv \{\psi \in \Gamma(\Sigma\mathbb{R}^n) : \psi(\gamma \cdot x) = \gamma \cdot \psi(x), \forall x \in \mathbb{R}^n, \forall \gamma \in \Gamma\},$$

where the  $\Gamma$ -action on both  $\mathbb{R}^n$  and  $\Sigma\mathbb{R}^n$  is denoted by “ $\cdot$ ”. The Dirac spectrum of  $T^n$  can now be computed explicitly.

**Theorem 4.1.1** (Dirac spectrum of flat tori). *For any positive integer  $n$ , the spectrum of the Dirac operator of the flat torus  $T^n$  endowed with the induced flat metric and the  $(\delta_1, \dots, \delta_n)$ -spin structure is given by*

$$\text{Spec}(D) = \left\{ \pm 2\pi \left| \gamma^* + \frac{1}{2} \sum_{i=1}^n \delta_i \gamma_i^* \right| : \gamma^* \in \Gamma^* \right\}$$

and each  $\gamma^*$  contributes multiplicity  $2^{\lfloor n/2 \rfloor - 1}$  if its corresponding eigenvalue is nonzero. In the case  $\delta_1 = \dots = \delta_n = 0$ , the multiplicity of the eigenvalue 0 is  $2^{\lfloor n/2 \rfloor}$ .

*Proof.* Note that the spinor bundle of  $\mathbb{R}^n$  is trivial, i.e. isomorphic to  $\mathbb{R}^n \times \Delta_n$ . For  $\psi \in \Gamma(\Sigma\mathbb{R}^n) \equiv C^\infty(\mathbb{R}^n, \Delta_n)$ , the equivariance condition reads

$$\psi(x + \gamma_k) = (-1)^{\delta_k} \psi(x)$$

for every  $x \in \mathbb{R}^n$  and  $1 \leq k \leq n$ . For each  $\gamma^* \in \Gamma^*$ , denote by  $\theta_\gamma$  the constant 1-form

$$\gamma^* + \frac{1}{2} \sum_{k=1}^n \delta_k \gamma_k^* \in (\mathbb{R}^n)^*.$$

Now, choose an arbitrary orthonormal basis  $(\sigma_l)_{1 \leq l \leq 2^{\lfloor n/2 \rfloor}}$  of  $\Delta_n$ , which can be extended to  $\mathbb{R}^n$  as elements of  $\Gamma(\mathbb{R}^n \times \Delta_n) \cong \Gamma(\Sigma\mathbb{R}^n)$ . Define the following spinor field over  $\mathbb{R}^n$ :

$$\phi_{\gamma,l} = e^{2i\pi\theta_\gamma} \sigma_l.$$

This spinor field satisfies the equivariance condition. To prove it, apply its definition,

$$\phi_{\gamma,l}(x + \gamma_k) = e^{2i\pi\theta_\gamma(x+\gamma_k)} \sigma_l(x + \gamma_k) = e^{2i\pi\theta_\gamma(\gamma_k)} \phi_{\gamma,l}(x).$$

Now, analyze the exponential term:

$$\gamma(\gamma_k) = \gamma^*(\gamma_k) + \frac{1}{2} \sum_{j=1}^n \delta_j \gamma_j^*(\gamma_k) = z + \frac{1}{2} \delta_k, \quad z \in \mathbb{Z},$$

so it follows that

$$e^{2i\pi\theta_\gamma(\gamma_k)} \phi_{\gamma,l}(x) = e^{i\pi\delta_k} \phi_{\gamma,l}(x) = (-1)^{\delta_k} \phi_{\gamma,l}(x).$$

Moreover, for any  $X \in \mathbb{R}^n$

$$\nabla_X \phi_{\gamma,l} = X(\phi_{\gamma,l}) = 2i\pi\theta_\gamma(X) e^{2i\pi\theta_\gamma} \sigma_l = 2i\pi\theta_\gamma(X) \phi_{\gamma,l},$$

so one can fix an orthonormal basis  $(e_1)_{1 \leq k \leq n}$  of  $\mathbb{R}^n$  and apply the Dirac operator,

$$D\phi_{\gamma,l} = \sum_{k=1}^n e_k \cdot \nabla_{e_k} \phi_{\gamma,l} = 2i\pi \sum_{k=1}^n \theta_\gamma(e_k) e_k \cdot \phi_{\gamma,l} = 2i\pi \theta_\gamma \cdot \phi_{\gamma,l}.$$

There are three separate cases that have to be analyzed. The first one is  $\theta_\gamma = 0$ , which only happens if  $\gamma^* = 0$  and  $\delta_1 = \dots = \delta_n = 0$ . Then, the constant spinor field  $\sigma_l$  is an eigenvector associated to the eigenvalue 0. Furthermore, since the torus is flat, i.e. its scalar curvature is zero, the Schrödinger-Lichnerowicz formula (3.3.1) guarantees that the kernel of  $D$  consists of parallel spinors. Hence,  $(\sigma_l)_{1 \leq l \leq 2^{\lfloor n/2 \rfloor}}$  generates the eigenspace  $E_0$ , so 0 is an eigenvalue with multiplicity  $2^{\lfloor n/2 \rfloor}$ .

If  $\theta_\gamma \neq 0$  and  $n = 1$ , then  $(i\theta_\gamma/|\theta_\gamma|) \cdot = \pm \text{id}$  on  $\Delta_1 = \mathbb{C}$ . Therefore  $D\phi_{\gamma,l} = \pm 2\pi|\theta_\gamma|\phi_{\gamma,l}$ . In other words,  $\phi_{\gamma,l}$  is a nonzero eigenspinor associated to  $2\pi|\theta_\gamma|$  or  $-2\pi|\theta_\gamma|$ . In either case, taking

$$\tilde{\gamma}^* = -\gamma^* - \sum_{k=1}^n \delta_k \gamma_k^*,$$

one gets

$$\theta_{\tilde{\gamma}} = -\theta_\gamma.$$

This ensures that both eigenvalues appear, so each eigenvalue has multiplicity 1.

The last case corresponds to  $\theta_\gamma \neq 0$ ,  $n \geq 2$ . As above, consider the operator  $P = (i\theta_\gamma/|\theta_\gamma|) \cdot$ . It is an involution (since  $\theta_\gamma^2 \cdot \psi = -|\theta_\gamma|^2 \psi$ ), and it is parallel, i.e.  $[P, \nabla] = 0$  (since  $\theta_\gamma$  is constant). Thus it induces the following orthogonal parallel splitting

$$\Delta_n = \Delta_{\text{id}} \oplus \Delta_{-\text{id}},$$

where  $\Delta_{\pm \text{id}} = \{\psi \in \Delta_n : P\psi = \pm \psi\}$  and both spaces have the same dimension. The latter claim is proved by taking a vector  $x$  orthogonal to  $\theta_\gamma$ . Clifford multiplication between two orthogonal vectors anti-commutes, so  $\Delta_{\pm \text{id}}$  corresponds to  $\Delta_{\mp \text{id}}$  by the bijection  $x \cdot$ .

As a result, the basis  $(\sigma_l)_{1 \leq l \leq 2^{\lfloor n/2 \rfloor}}$  can be replaced by

$$(\sigma_1^+ + \dots + \sigma_{2^{\lfloor n/2 \rfloor - 1}}^+, \sigma_1^- + \dots + \sigma_{2^{\lfloor n/2 \rfloor - 1}}^-),$$

where  $(\sigma_1^\pm + \dots + \sigma_{2^{\lfloor n/2 \rfloor - 1}}^\pm)$  is a constant orthonormal basis of  $\Delta_{\pm \text{id}}$ . Changing the notation from  $\phi_{\gamma,l}^\pm$ , the same computations apply, so

$$D\phi_{\gamma,l}^\pm = 2i\pi \theta_\gamma \cdot \phi_{\gamma,l}^\pm = 2i\pi(\mp i|\theta_\gamma|) \phi_{\gamma,l}^\pm = \pm 2\pi|\theta_\gamma| \phi_{\gamma,l}^\pm.$$

This equation implies that  $\phi_{\gamma,l}^\pm$  is a nonzero eigenvector of  $D$ , associated to the eigenvalue  $\pm 2\pi|\theta_\gamma|$ . Since the spinor fields  $\phi_{\gamma,1}^\pm, \dots, \phi_{\gamma,2^{\lfloor n/2 \rfloor - 1}}^\pm$  are linearly independent, the multiplicity of each eigenvalue is at least  $2^{\lfloor n/2 \rfloor - 1}$ .

The proof is almost complete. The only point left to explain is that the set  $\{e^{i\gamma^*}, \gamma^* \in \Gamma^*\}$  is a Hilbert basis of  $L^2(T^n, \mathbb{C})$  [20], so the set of spinor fields of the form  $\phi_{\gamma, l}^\pm$  is also a Hilbert basis of  $L^2(\Sigma T^n)$ . This guarantees that no more eigenspinors exist. ■

*Remark 4.1.2.* The reader should be aware of the fact that each eigenspinor has at least one analogue with which it shares eigenvalues. Indeed, taking

$$\bar{\gamma}^* = -\gamma^* - \sum_{k=1}^n \delta_k \gamma_k^* \in \Gamma^*$$

provides the same eigenvalue as  $\gamma^*$ , but  $\gamma^* \neq \bar{\gamma}^*$ . Therefore the multiplicity of every eigenvalue is at least  $2 \cdot 2^{\lfloor n/2 \rfloor - 1} = 2^{\lfloor n/2 \rfloor}$ .

**Example 4.1.3** (The circle as a 1-torus). For  $n = 1$ , the theorem gives the Dirac spectrum for 1-spheres of length  $L > 0$ ,  $S^1(L)$ . Indeed, take  $\gamma_1 = L$ ,  $\gamma_1^* \equiv 1/L$ . With this notation, every  $\gamma^*$  is of the form  $\mathbb{Z}/L$ , so it follows that

$$\text{Spec}(D) = \frac{2\pi}{L} \left( \frac{\delta}{2} + \mathbb{Z} \right), \quad \delta \in \{0, 1\},$$

where the value of  $\delta$  fixes one of the  $2^1 = 2$  possible spin structures.

### A brief note on Bieberbach manifolds

Although the computation of the Dirac spectrum is not particularly complicated in the case of flat tori, this changes drastically when considering general compact connected flat manifolds, also known as “Bieberbach manifolds” after the mathematician that proved that every such manifold  $M$  is covered by a flat torus  $T^n$ . This ensures that spinors on  $M$  correspond to those on  $T^n$  satisfying a certain equivariance condition. The Dirac spectrum of  $M$  is therefore contained in that of  $T^n$ .

Nonetheless, the equivariance condition is rather technical, and explicit computation of the Dirac spectrum on general Bieberbach manifolds is so challenging that only dimension 3 (F. Pfäffle [21]) and some particular higher dimension cases (R. Miatello and R. Podestá [22]) have been successfully handled via representation theory. Their results are at a level of complexity well above that of this work, so trying to go into more detail here seems rather hopeless.

### 4.1.2 Spherical space forms

A space form is a complete Riemannian manifold of constant sectional curvature. In particular, spherical space forms are connected Riemannian manifolds locally isometric to the  $n$ -sphere  $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ , endowed with its canonical metric of sectional curvature 1 (see [23] for a classification of spherical space forms). In fact, the first step in the determination of their spectra is calculating the spectrum of  $S^n$ .

Other space forms are quotients  $M = \Gamma \backslash S^n$  by the action of suitable finite groups  $\Gamma \subset \mathrm{SO}_n$ . They are spin only if  $n$  is odd and there is a group homomorphism  $\epsilon : \Gamma \rightarrow \mathrm{Spin}_{n+1}$  such that  $\tilde{\rho} \circ \epsilon = \mathrm{id}_\Gamma$ , where  $\tilde{\rho} : \mathrm{Spin}_{n+1} \rightarrow \mathrm{SO}_{n+1}$  is the double covering mapping [24].

If  $M$  is spin, eigenspinors of  $D$  on it correspond to  $\epsilon(\Gamma)$ -invariant eigenspinors on the sphere, so all eigenvalues of  $M$  are also eigenvalues of  $S^n$ , only with generally smaller multiplicities.

#### Dirac spectrum of the $n$ -sphere

There are several methods to determine the spectrum of  $S^n$ ,  $n \geq 2$ , but the most elemental one, developed by C. Bär [24], makes use of the notion of Killing spinors. It is known [25] that the dimension of the space of 1/2-Killing spinors is  $2^{\lfloor n/2 \rfloor}$ . The same is true for the space of  $-1/2$ -Killing spinors. As a result,  $\Sigma S^n$  can be trivialized through one kind of Killing spinors or the other. It also uses the following lemma regarding the scalar Laplace spectrum on  $S^n$ .

**Lemma 4.1.4.** *The eigenvalues of the scalar Laplacian on  $S^n$  are*

$$k(n+k-1) \quad k = 0, 1, 2, \dots$$

*and each of them has multiplicity*

$$m_k = \binom{n+k-1}{k} \frac{n+2k-1}{n+k-1}.$$

*Proof.* See [26, pp. 159-162]. ■

All the pieces are in place for the following theorem.

**Theorem 4.1.5** (Dirac spectrum of  $n$ -spheres). *Let  $n \geq 2$ . The spectrum of the Dirac operator of  $S^n$  is*

$$\mathrm{Spec}(D) = \left\{ \pm \left( \frac{n}{2} + k \right) : k = 0, 1, 2, \dots \right\},$$

and each eigenvalue has multiplicity

$$m_k = \binom{n+k-1}{k} 2^{\lfloor n/2 \rfloor}.$$

*Proof.* Let  $\psi$  be a  $\pm 1/2$ -Killing spinor. Then it is clear that

$$D\psi = \sum_{i=1}^n e_i \cdot \nabla_{e_i} \psi = \pm \frac{1}{2} \sum_{i=1}^n e_i \cdot e_i \cdot \psi = \mp \frac{n}{2} \psi.$$

Now, for every  $f \in C^\infty(S^n, \mathbb{R})$ , use Lemma 3.2.1 to find

$$\begin{aligned} D^2(f\psi) &= fD^2\psi - 2\nabla_{\text{grad}(f)}\psi + (\Delta f)\psi = \frac{n^2}{4}f\psi \mp \text{grad}(f) \cdot \psi + (\Delta f)\psi \\ &= \frac{n^2}{4}f\psi \mp (D(f\psi) - fD\psi) + (\Delta f)\psi = \left(\frac{n^2}{4} - \frac{n}{2}\right)f\psi \mp D(f\psi) + (\Delta f)\psi. \end{aligned}$$

Rearranging terms yields

$$\left(D \pm \frac{1}{2}\right)^2 (f\psi) = \left[\Delta f + \frac{(n-1)^2}{4}f\right]\psi.$$

Let  $\{f_k\}_k$  be an  $L^2$ -orthonormal basis of eigenfunctions of  $\Delta$ , and  $\{\psi_i\}_{1 \leq i \leq 2^{\lfloor n/2 \rfloor}}$  a trivialization of  $\Sigma S^n$  by  $\pm 1/2$ -Killing spinors. Then, the set

$$\left\{f_k \psi_i : k = 0, 1, \dots, 1 \leq i \leq 2^{\lfloor n/2 \rfloor}\right\}$$

is a complete orthonormal basis of  $L^2(\Sigma S^n)$ , consisting of eigenspinors of  $(D \pm 1/2)^2$ , associated to the eigenvalues

$$k(n+k+1) + \frac{(n-1)^2}{4} = \left(k + \frac{n-1}{2}\right)^2,$$

where Lemma 4.1.4 has been used. The multiplicities are computed by combining the cardinal of the Killing spinor basis with the Laplacian multiplicities,

$$m_k = 2^{\lfloor n/2 \rfloor} \binom{n+k-1}{k} \frac{n+2k-1}{n+k-1}.$$

As a result, the following inclusion holds:

$$\text{Spec}(D) \subset \left\{ \mp \frac{1}{2} \pm \left( \frac{n-1}{2} + k \right) : k = 0, 1, \dots \right\},$$

hence there are four different kinds of possible eigenvalues for  $D$ :

$$\begin{aligned} \lambda_k^+ &:= \frac{n}{2} + k, & \lambda_{-(k+1)}^+ &:= -\frac{n}{2} - k + 1, \\ \lambda_k^- &:= -\frac{n}{2} - k, & \lambda_{-(k+1)}^- &:= \frac{n}{2} + k - 1. \end{aligned}$$

Denote by  $m(\cdot)$  their corresponding multiplicity, where it is clear that

$$m(\lambda_k^\pm) + m(\lambda_{-(k+1)}^\pm) = m_k. \quad (4.1.1)$$

The last step is determining the multiplicity of  $\lambda_k^\pm$ . This is done by induction on  $k$ .

**Claim.**  $m(\lambda_k^\pm) = 2^{\lfloor n/2 \rfloor} \binom{n+k-1}{k} \quad (k = 0, 1, \dots)$ .

Indeed, for  $k = 0$ ,  $\lambda_0^\pm$  both have multiplicity  $2^{\lfloor n/2 \rfloor}$ , since  $\pm 1/2$ -Killing spinors are eigen-spinors for  $D$ , associated to the eigenvalue  $\mp n/2$ . On the other hand,  $\lambda_{-1}^\pm = \mp(n/2 - 1)$  cannot appear as a result of Friedrich's inequality (see Section 4.2.1). Hence assume the claim holds for  $k$ , then

$$\begin{aligned} m(\lambda_{k+1}^\pm) &\stackrel{(4.1.1)}{=} 2^{\lfloor n/2 \rfloor} \binom{n+k}{k+1} \frac{n+2k+1}{n+k} - m(\lambda_{-(k+2)}^\pm) \\ &= 2^{\lfloor n/2 \rfloor} \binom{n+k}{k+1} \frac{n+2k+1}{n+k} - m(\lambda_k^\mp) \\ &= 2^{\lfloor n/2 \rfloor} \left[ \binom{n+k}{k+1} \frac{n+2k+1}{n+k} - \binom{n+k-1}{k} \right] = 2^{\lfloor n/2 \rfloor} \binom{n+k}{k+1}, \end{aligned}$$

so the claim holds for every  $k = 0, 1, \dots$ . This completely characterizes the spectrum, thus completing the proof.  $\blacksquare$

### Dirac spectrum of spin spherical space forms

As has already been established, the Dirac spectrum of general spin spherical space forms is a subset of that of  $S^n$ . The only variable left to determine is the multiplicity of each eigenvalue. C. Bär has also shown that this information is encoded in the following formal power series:

$$F_\pm(z) = \sum_{k=0}^{\infty} m \left[ \pm \left( \frac{n}{2} + k \right) \right] z^k, \quad (4.1.2)$$

which converge absolutely for  $|z| < 1$ . The following theorem, which will be cited without proof, gives formulas for  $F_\pm(z)$  in terms of the group  $\Gamma$  and the group homomorphism  $\epsilon$ . The idea is that, if this new formula can be turned into a power series of the form (4.1.2), then the spectrum is characterized by matching both series term by term.

**Theorem 4.1.6** (C. Bär [24]). *Let  $\Gamma \backslash S^n$ , with  $n$  odd, be a spherical space form with a spin structure fixed by  $\epsilon : \Gamma \rightarrow \text{Spin}_{n+1}$ . Then,*

$$\text{Spec}(D) \subset \left\{ \pm \left( \frac{n}{2} + k \right) : k = 0, 1, 2, \dots \right\}$$

and  $F_\pm(z)$  are determined by

$$F_\pm(z) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{\chi^\mp(\epsilon(\gamma)) - \chi^\pm(\epsilon(\gamma)) \cdot z}{\det(\text{id}_{\mathbb{R}^{n+1}} - z \cdot \gamma)},$$

where  $\chi^\pm := \text{tr}(\delta_{n+1}^\pm) : \text{Spin}_{n+1} \rightarrow \mathbb{C}$  is the character of  $\delta_{n+1}^\pm$ . Thus the multiplicities of the eigenvalues are determined by the coefficients of every  $z^k$  in the series expression around

zero of the holomorphic functions  $F_{\pm}(z)$ .

Among other applications, this theorem allows for explicit calculation of the Dirac spectrum of spin real projective spaces  $\mathbb{RP}^n \equiv \mathbb{Z}_2 \backslash S^n$  [13, p. 34].

Flat tori and spherical space forms are not the only manifolds that admit an explicit, at least partial, calculation of the Dirac spectrum. Other examples include some homogeneous and symmetric spaces. However, this work shall not go further into them. An interested reader can find more information in the thorough, albeit rather novice-unfriendly, review by N. Ginoux [13, Chapter 2].

## 4.2 Lower nonzero eigenvalue bounds

Even though explicit calculations are often rather unfeasible for manifolds other than very basic examples, that does not mean that study of the Dirac spectrum is a fool's errand. Many different estimations for its lower bounds exist (see, for example, [14] or [27]), with varying levels of complexity and sharpness. While explicit computations require a very specific kind of manifold, these bounds usually only need relatively mild assumptions.

Upper bound results for specific eigenvalues also exist, but they are more convoluted in nature than the ones that will be presented here. The reader is referred once again to N. Ginoux's book [13, Chapter 5].

In this last section of the survey, the most elemental lower eigenvalue bounds for closed manifolds are presented. A brief commentary on compact manifolds with boundary is also provided, although explicit calculations are avoided in this case.

### 4.2.1 Bounds on closed manifolds

Closed manifolds provide a wide range of tools to obtain some lower eigenvalue estimates via fairly straightforward reasonings. The most elemental of them arises as a natural consequence of the Schrödinger-Lichnerowicz formula (Theorem 3.3.5).

**Theorem 4.2.1.** *On a closed Riemannian spin manifold  $(M^n, g)$  with positive scalar curvature  $S$ , the Dirac operator satisfies*

$$(i) \quad \ker(D) = \{0\}.$$

(ii) If  $\lambda \in \mathbb{R}$  is a nonzero eigenvalue of  $D$ , then

$$\lambda^2 > \frac{1}{4}S_0, \quad S_0 := \inf_M(S).$$

*Proof.* Consider a nontrivial spinor field  $\psi \in \Gamma(\Sigma M)$ . By the Schrödinger-Lichnerowicz formula, one can write

$$\int_M \langle D^2\psi, \psi \rangle \nu_g = \int_M \langle \nabla^* \nabla \psi, \psi \rangle \nu_g + \int_M \frac{1}{4} S \langle \psi, \psi \rangle \nu_g.$$

By definition,  $\nabla^*$  is the formal adjoint of  $\nabla$ , and since  $D$  is formally self-adjoint,

$$\int_M |D\psi|^2 \nu_g = \int_M |\nabla \psi|^2 \nu_g + \int_M \frac{1}{4} S |\psi|^2 \nu_g. \quad (4.2.1)$$

Since  $S > 0$ ,  $D\psi$  cannot be identically zero, so property (i) is satisfied. To prove property (ii), choose an eigenspinor  $\psi$  associated to the eigenvalue  $\lambda$ . Rearranging the previous equation yields

$$\int_M |D\psi|^2 \nu_g - \int_M \frac{1}{4} S |\psi|^2 \nu_g = \int_M |\nabla \psi|^2 \nu_g \geq 0.$$

Hence

$$\lambda^2 - \frac{1}{4}S_0 \geq 0,$$

where the equality only holds if  $\nabla \psi = 0$ . However, this would imply  $D\psi = 0$ , which is forbidden by (i). Therefore the inequality is strict.  $\blacksquare$

### Friedrich's inequality

The fact that the above inequality is strict is noteworthy. A first step toward sharper bounds could be trying to obtain one which allowed for the equality to be achieved. Indeed, this is what German mathematician T. Friedrich achieved with the following inequality, which bears his name.

**Theorem 4.2.2** (Friedrich's inequality). *On a closed Riemannian spin manifold  $(M^n, g)$  ( $n \geq 2$ ), any eigenvalue  $\lambda \in \mathbb{R}$  of the Dirac operator satisfies*

$$\lambda^2 \geq \frac{n}{4(n-1)} S_0.$$

*Proof.* It is obvious that the inequality holds if  $S_0 \leq 0$ , so only the case  $S > 0$  has to be explored. Choose an arbitrary  $\psi \in \Gamma(\Sigma M)$  and a local orthonormal basis  $(e_i)_{1 \leq i \leq n}$ . Then,



using the Cauchy-Schwarz (CS) inequality,

$$\begin{aligned} |D\psi|^2 &= \left| \sum_{i=1}^n e_i \cdot \nabla_{e_i} \psi \right|^2 \leq \left( \sum_{i=1}^n |e_i \cdot \nabla_{e_i} \psi| \right)^2 \\ &\stackrel{(3.1.1)}{=} \left( \sum_{i=1}^n |\nabla_{e_i} \psi| \right)^2 \stackrel{\text{CS}}{\leq} n \sum_{i=1}^n |\nabla_{e_i} \psi|^2 = n |\nabla \psi|^2. \end{aligned}$$

Combining this with (4.2.1), one gets

$$\frac{1}{n} \int_M |D\psi|^2 \nu_g \leq \int_M |D\psi|^2 \nu_g - \int_M \frac{1}{4} S |\psi|^2 \nu_g,$$

or, rearranging the integrals,

$$\left(1 - \frac{1}{n}\right) \int_M |D\psi|^2 \nu_g \geq \int_M \frac{1}{4} S |\psi|^2 \nu_g.$$

For  $\psi \in \Gamma(\Sigma M)$  such that  $D\psi = \lambda\psi$ , one has

$$\lambda^2 \int_M |\psi|^2 \nu_g \geq \frac{n}{4(n-1)} \int_M S |\psi|^2 \nu_g \geq \frac{n}{4(n-1)} S_0 \int_M |\psi|^2 \nu_g.$$

Friedrich's inequality trivially follows. ■

### Hijazi's inequality

Friedrich's inequality is not impervious to improvements. There are several ways one can go about finding new lower eigenvalue bounds, but the one that will be presented here makes use of the conformal covariance of the Dirac operator (see Section 3.4). This method is owed to O. Hijazi.

**Theorem 4.2.3** (Hijazi's inequality). *Let  $(M^n, g)$  be a closed Riemannian spin manifold with  $n \geq 2$  and  $u \in C^\infty(M, \mathbb{R})$ . Let  $\bar{g} := e^{2u}g$  be a conformally transformed metric and denote the scalar curvature of  $(M^n, \bar{g})$  by  $\bar{S}$ . Then any  $\lambda \in \text{Spec}(D)$  satisfies*

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M (\bar{S} e^{2u}).$$

*Proof.* Recall the equation given by Proposition 3.4.1:

$$\bar{D} \left( e^{-(n-1)u/2} \bar{\psi} \right) = e^{-(n+1)u/2} \bar{D} \bar{\psi}.$$

As a result of it, taking  $\phi := e^{-(n-1)u/2} \psi$  implies

$$D\psi = \lambda\psi \implies \bar{D}\bar{\phi} = \lambda e^{-u} \bar{\phi}.$$

Now, since the Penrose operator can also be conformally transformed (Proposition 3.4.4), it is possible to take the decomposition  $|\bar{P}\bar{\phi}|^2 = |\bar{\nabla}\bar{\phi}|^2 - |\bar{D}\bar{\phi}|^2/n$ . Combining this with

the Schrödinger-Lichnerowicz integral (4.2.1) one gets a new integral equality:

$$\int_M |\bar{P}\bar{\phi}|^2 \nu_{\bar{g}} = \frac{n-1}{n} \int_M |\bar{D}\bar{\phi}|^2 \nu_{\bar{g}} - \frac{1}{4} \int_M \bar{S} |\bar{\phi}|^2 \nu_{\bar{g}}.$$

Suppose that  $\phi$  comes from an eigenspinor of  $D$  associated to the eigenvalue  $\lambda$ . Rearranging terms,

$$\frac{n}{n-1} \int_M |\bar{P}\bar{\phi}|^2 \nu_{\bar{g}} = \int_M \left[ \lambda^2 - \frac{n}{4(n-1)} \bar{S} e^{2u} \right] e^{-2u} |\bar{\phi}|^2 \nu_{\bar{g}} \geq 0. \quad \blacksquare$$

Many more lower eigenvalue bounds exist. Methods for finding improvements on Friedrich's inequality are varied, from using parallel forms to introducing the energy-momentum tensor, which is a concept that any physicist would immediately recognize. Nevertheless, for the purposes of this review, Hijazi's inequality is enough to illustrate how sharper bounds can be determined.

#### 4.2.2 Brief note on bounds on manifolds with boundary

The case of general compact Riemannian spin manifolds with boundary is far more involved than that of closed manifolds. The principal reason for this is that now, boundary conditions must be taken into account. In particular, the most interesting boundary conditions are those called “elliptic”, which are pseudo-differential operators  $B : L^2(\Sigma\partial M) \rightarrow L^2(V)$ , where  $V$  is some Hermitian vector bundle over  $\partial M$ ; that guarantee certain existence and smoothness solutions to the boundary value problem

$$\begin{cases} D\psi = \varphi & \text{on } M, \\ B(\psi|_{\partial M}) = \chi & \text{on } \partial M. \end{cases}$$

In fact, self-adjoint elliptic conditions restrict the Dirac spectrum to real values (see [13, p. 23ff.] for more information).

There are uncountably many of these conditions, the four most often invoked being the usual and modified Atiyah-Patodi-Singer boundary conditions (gAPS and mgAPS), the conditions associated to a chirality operator (CHI), and the MIT boundary condition. Each one of them presents unique kinks, advantages and disadvantages. One of their many applications, and the one most closely related to this survey, is the fact that under any of them, some kind of modified Friedrich's inequality holds. As an example to close out this survey, take the following theorem found in [13, p. 69].

**Theorem 4.2.4.** *Let  $n \geq 2$  and  $(M^n, g)$  be a compact Riemannian spin manifold with nonempty boundary  $\partial M$ . If  $\partial M$  has nonnegative mean curvature with respect to the inner*

normal, then any  $\lambda \in \text{Spec}(D)$  under the *gAPS* boundary condition satisfies

$$\lambda^2 > \frac{n}{4(n-1)} S_0.$$

From this theorem, it is inferred that some boundary conditions can impose harsher bounds on the Dirac spectrum, since now the equality cannot be attained. Nonetheless, different boundary conditions yield different result, so they are a source of a highly varied and creative study of the Dirac operators.



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